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WITH THEIR

SOLUTIONS.

FROM THE "EDUCATIONAL TIMES."

VOL. LXXV.

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WITH MANY

PAPERS AND SOLUTIONS

IN ADDITION TO THOSE

PUBLISHED IN THE "EDUCATIONAL TIMES."

EDITED BY

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$$= \frac{1}{\Gamma(m)\,\Gamma(n)} \int_0^1 \int_0^1 e^{xy} (1-x)^{n-1} (1-y)^{m-1} x^{m-n+1}\, dx\, dy \quad \dots\dots 65$$

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13237. (R. W. D. Christie.)—P, Q, R act at a point parallel to the sides taken in order of an equilateral triangle, and the resultant is b^2+c^2-bc. Show that, while P is arbitrary, Q and R have each three separate values, and find them 35

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14198. (Professor Sanjána, M.A.)—If p_1, p_2, p_3 be the parameters of the three parabolas which respectively touch one side of a triangle at its mid-point and the other two sides produced, then

$$\Sigma p^{-\frac{1}{2}} = \tfrac{3}{2}\frac{\cot\omega}{(2\Delta)^{\frac{1}{2}}} \quad \text{and} \quad \Sigma p^{-\frac{1}{2}} = \tfrac{9}{8}\frac{\cot^2\omega - 1}{(2\Delta)^{\frac{1}{2}}}.$$

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14199. (Professor Umes Chandra Ghosh.) — ACB is a quadrant of an ellipse, and CP any central radius vector. If x and y are the distances of the centre of gravity of the segment BCP of the ellipse from the major and minor axes, show that

$$ay = bx\cot\left\{\frac{2ab^2x}{3(a^2y^2+b^2x^2)}\right\} \text{ ..... } 78$$

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14233. (Professor K. J. Sanjána.) — If three points A, B, C be referred to two axes meeting in O, the intersections of the joins of B and C, C and A, A and B, with the isogonal conjugates with regard to the given axes of OA, OB, OC, respectively, are collinear. If three straight lines BC, CA, AB be referred to two points on a line Δ, the joins of A, B, C with the isotomic conjugates with regard to the given points of the intersections of Δ with BC, CA, AB, respectively, are concurrent 28

14277. (Robert W. D. Christie.) — Find (1) a general value for x in the equation $3.2^x+1=pq$ when p is a prime. A few examples are

$x = 3n-2$ (n any integer); then $p=7$ (*v.* Quest. 14158)
$= 4n-1$,, $p=5$
$= 10n-3$,, $p=11$,

and (2) do the same for $3.2^x-1=pq$ 52

14296. (H. MacColl, B.A.) — If x, y, z be each taken at random between 0 and 1, show that the chance that the fraction $\frac{z(1-x-y)}{1-y-yz}$ will also be between 0 and 1 is $\frac{5}{4}-\log_e 2$, which is very nearly 5/9 38

14320. (J. J. Barniville, B.A.)—Prove that

$$\frac{1}{1+2}+\frac{1}{13+2}+\frac{1}{233+2}+\ldots=\frac{\sqrt{5}+1}{8};$$

$$\frac{1}{2+2}+\frac{1}{34+2}+\frac{1}{610+2}+\ldots=\frac{\sqrt{5}}{8};$$

$$\frac{1}{5+2}+\frac{1}{89+2}+\frac{1}{1597+2}+\ldots=\frac{\sqrt{5}-1}{8};$$

$$\frac{1}{1+5}+\frac{1}{2+5}+\frac{1}{5+5}+\frac{1}{13+5}+\ldots=\frac{5\sqrt{5}}{22};$$

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14369. (G. H. Hardy, B.A. — If $\sum_0^\infty u_n(x)$ be a *summable divergent* series, having the sum $\int_0^\infty e^{-\alpha}u(x,\alpha)\,d\alpha$,

where $u(x,\alpha)=\sum_0^\infty \frac{\alpha^n}{n!}u(x)$,

prove that it may be integrated term by term in (x_0, X), provided (1) $u(x,\alpha)$ may be integrated term by term, and (2) the order of integration in $\int_0^\infty e^{-\alpha}d\alpha\int_{x_0}^X u(x,\alpha)\,dx$

may be changed. Deduce and verify independently that, if

$$\phi(x,\alpha)=e^{-\alpha(1+\cos x)}\cos(x-\alpha\sin x),$$
$$\psi(x,\alpha)=e^{-\alpha(1-\cos 2mx)}\cos(mx+\alpha\sin 2mx)/(1+x^2),$$

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then $\int_0^{2\pi} dx \int_0^{\infty} \phi\, d\alpha = \pi, \quad \int_0^{\infty} d\alpha \int_0^{2\pi} \phi\, dx = 0;$

$\int_0^{\infty} dx \int_0^{\infty} \psi\, d\alpha = 0, \quad \int_0^{\infty} d\alpha \int_0^{\infty} \psi\, dx = \frac{1}{2} \frac{\pi}{e^{m} - e^{-m}}$ 135

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$$\frac{1}{1^2+3} + \frac{1}{5^2+3} + \frac{1}{19^2+3} + \frac{1}{71^2+3} + \ldots = \frac{1}{2\sqrt{3}};$$

$$\frac{1}{1^2+8} + \frac{1}{5^2+8} + \frac{1}{19^2+8} + \frac{1}{71^2+8} + \ldots = \frac{1}{4\sqrt{3}};$$

$$\frac{1}{5^2-1} + \frac{1}{19^2-1} + \frac{1}{71^2-1} + \frac{1}{265^2-1} + \ldots = \frac{1}{3} - \frac{1}{2\sqrt{3}};$$

$$\frac{1}{5^2-6} + \frac{1}{19^2-6} + \frac{1}{71^2-6} + \frac{1}{265^2-6} + \ldots = \frac{1}{5} - \frac{1}{4\sqrt{3}}$$ 39

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14422. (Professor E. Lemoine.)—Soient deux triangles ABC, A′B′C′ dont j'appelle les côtés a, b, c ; a', b', c'. Je suppose ABC fixe. Ils sont tels que a et a' sont perpendiculaires ainsi, que b et b' et que c et c'. Cela posé, si ABC et A′B′C′ sont homologiques, le lieu du centre O d'homologie est une conique D circonscrite à ABC ; si ABC et B′C′A′ sont homologiques, le lieu est une conique D′ circonscrite à ABC ; si ABC et C′A′B′ sont homologiques, le lieu est une conique D″ circonscrite à ABC. Les trois coniques D, D′, D″ ont pour point commun le point de Tarry du triangle ABC. Déterminer le triangle A′B′C′ qui fait en même temps des centres d'homologie O, O′, O″ avec ABC, c'est à dire que ABC et A′B′C′ soient trihomologiques. Remarquons que par leur construction ABC et A′B′C′ sont toujours triorthologiques 74

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R being the radius of the circumcircle of the triangle, B the origin, and BC the axis of x 56

14489. (Professor N. Bhattacharyya.) — A triangle ABC is such that its circumcentre is coincident with the centre of its Brocard ellipse. The lines AΩ, BΩ, CΩ meet the opposite sides in D, E, F, respectively ; and likewise the lines AΩ′, BΩ′, CΩ′ meet the opposite sides in D′, E′, F′, respectively. Let the lines EF, E′F′ ; FD, F′D′ ; DE, D′E′ meet, produced, if necessary, in P, Q, R, respectively. Prove that (1) the lines AP, BQ, CR are concurrent; (2) the sum of the squares of the axes of the ellipse $=4(5R^2-2\Delta\cot\varpi)$, where R denotes the circumradius, Δ the area, and ϖ the Brocard angle of the triangle 104

14521. (D. Biddle.)—[In continuation of Quests. 14457 and 14490.] Writing down N in terms of 3 and its powers, the only coefficients used being ±1 or 0, it will be convenient to omit both index and root, using only the sign preceding (or 0), and reckoning the powers from the right hand (as usual). Thus, $1843=3^7-3^5-3^4+3^2-3+1$, and it may be written $+0--0+-+$. Similarly, 1769 may be written $+-++0---$. Since the last sign is for $3^0(=1)$, the first index is 1 less than the number of signs (including zero). Prove that, having bracketed together any number of the given signs beginning with the first (which is always positive), and having assigned to them the value they would have if

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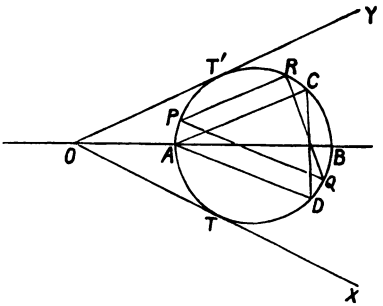
standing alone, we can, to the successive signs beyond, append p, p^2, p^3, ... in ascending order, and thus obtain a multiple of $3p \pm 1$, a factor of N (p being even), provided we reverse the signs attached to the odd powers of p when the factor is $3p+1$, but leave them unchanged when the factor is $3p-1$. Thus, $1769 = (+ - + + 0) - - -$, and $66 + 20 - 400 + 8000 = M(61)$; also $66 - 10 - 100 - 1000 = M(29)$ 49

14523. (R. Tucker, M.A.) — Pairs of chords are drawn, from points on a given circumference, in fixed directions. Find the locus of the centroids of the variable triangles (see figure) ... 30



14530. (Robert W. D. Christie.)—If $6m+1$ = a prime, prove

$$r^{2m} + r_1^m \equiv 0 \pmod{p} \quad \text{and} \quad r_1^{2m} + r^m \equiv 0 \pmod{p},$$

where r and r_1 are associated primitive roots 47

14533. (Rev. T. Wiggins, B.A.)—Express the numbers 131, 133, 137, 139 by four nines. Algebraical symbols and decimal points may be used, but the expression for each number must contain the four nines *and no other figure*. Also, express the number 113 by four fours, with similar conditions. (The above is part of Quest. 6487, which was reproposed in *Ed. Times*, March, 1900) 35

14535. (R. Knowles.) — A circle touches a parabola in a point P, and cuts it in C, D; prove (1) that the line joining the pole of CD, with respect to the parabola, to P is bisected by the axis; (2) if the circle pass through the focus, CD passes through a fixed point on the axis 129

14552. (D. Biddle.)—It being given that $N = 2\Delta M + 1$, and that its factors are of the form $2\Delta m + 1$, prove the validity of the method of factorization employed in the adjoining example, the rule being as follows:—Take, as first divisor (D_1), the number next below $\sqrt{N}$ of proper form, say $2\Delta m_1 + 1$; and take, as first dividend, $M - m_1$. These yield, as quotient and remainder, Q_1 and R_1. Since divisors patently composite can be discarded, take, in succession, $D_n = D_{n-1} - 2\Delta\mu_n$, where $\mu_n = 1$ or other small digit that does not carry D_n beyond the next prime (of the particular form) in descending order. Next, utilize

$N = 150809, \quad \Delta = 7.$

10745	(μ)	(κ)	(Q)	(R)
379			28	133
337	3	3	31	301
323	1	2	33	90
281	3	5	38	74
253	2	4	42	128
239	1	3	45	0

$2 \cdot 7 \cdot 45 + 1 = 631;$

$\therefore \quad N = 239 \cdot 631.$

$$(2\Delta\mu_n Q_{n-1} + R_{n-1} + \mu_n) / D_n = \kappa_n$$

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with R_n; also $\kappa_n + Q_{n-1} = Q_n$. When $R_x = 0$, $N = D_x(2\Delta Q_x + 1)$. [In the example, two divisors are tried which might have been excluded, namely, $323 = 17 \cdot 19$ and $253 = 11 \cdot 23$; but the final result is unaffected thereby] 48

14553. (R. Tucker, M.A.)—AP, AQ are vertical vectors of a parabola, and PQ is a focal chord. The circle APQ cuts the curve in R. The conormal point (of PQR) lies on a parabola, as also does the circumcentre of APQ. The intersections of (AP, QR) and of (AQ, PR) lie on the same cubic, and PR touches a semi-cubical parabola. AR, PQ intersect on a parallel to the directrix 96

14556. (R. P. Paranjpye, B.A.)—Trace the curve $(x/a)^n + (y/b)^n = 1$ when n is an infinite positive integer, distinguishing the cases when it is odd or even 35

14558. (Rev. J. Cullen.)—Show that the method of Quest. 14506 can be used to determine the prime or composite character of all numbers of the form $4n+1$ of eleven figures (or under). Apply the method to express $3 \cdot 2^{30} + 1$ as the sum of two squares and to determine if this partition is unique 102

14559. (Lt.-Col. Allan Cunningham, R.E.)—Solve

$$s_x = s_a \cdot s_y, \quad s_a \cdot s_x = s_a \cdot s_y, \quad s_a \cdot s_\beta \cdot s_\gamma \ldots s_x = s_a \cdot s_b \cdot s_c \ldots s_y,$$

where s denotes a sum of *successive* integer cubes 120

14560. (Robert W. D. Christie.)—

A gentleman a garden had
Of a yards long and c yards broad.
A walk of equal width all round
He made that took one-fourth the ground.
Ye skilful in geometry,
Tell me how wide the walk must be.
All fractions discard and integers use,
Or time and trouble you'll certainly lose 33

14564. (R. Chartres.)—If ABC be a triangle, P and Q random points, find the mean value of the nth power of the triangle PAQ (1) when P and R are confined to the base BC, (2) when P and Q can be anywhere in the triangle. Also find the same when ABC is a sector of a circle, centre A; and deduce the former results from these 29

14565. (Salutation.)—AB, CD are diameters, perpendicular to each other, of the circle ACBD; P is a point on the circumference between A and C; and, with centre A and radius AP, the arc PQR is described, meeting the circle again in R, and having CQ tangential to it. Prove that the arc PQR bisects the circle, when the radius of the latter bears to that portion of its circumference which lies between P and B the same ratio that subsists between CQ^2 and the area (of the circle) lying outside the rectangle APBS 83

14566. (R. J. Dallas, M.A.)—If we have a set of $n-1$ equations

$$a(1 + x_r^2 x_{r+1}^2) - b(x_r^2 + x_{r+1}^2) + c x_r x_{r+1} = 0 \quad (r = 1, 2, \ldots, n-1),$$

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these cannot coexist with the equation $a(1+x_n^2x_1^2)-b(x_n^2+x_1^2)+cx_nx_1=0$ unless a certain relation holds between a, b, c when the equations are indeterminate. Find these relations for the cases $n=3, 4, 5, 6$ ... 99

14569. (John C. Malet, F.R.S.)—Being given that the roots of an algebraic equation of the twelfth degree, α_1, α_2, α_3, α_4, β_1, β_2, β_3, β_4, γ_1, γ_2, γ_3, γ_4, are connected by the relations

$$\alpha_1+\alpha_2+\alpha_3+\alpha_4=\beta_1+\beta_2+\beta_3+\beta_4=\gamma_1+\gamma_2+\gamma_3+\gamma_4,$$
$$\alpha_1\alpha_2\alpha_3\alpha_4=\beta_1\beta_2\beta_3\beta_4=\gamma_1\gamma_2\gamma_3\gamma_4,$$

solve the equation 25

14572. (Professor E. Lemoine.)—Soit ABC un triangle. Appelons A_1, B_1, C_1 les pieds sur BC, CA, AB des trois bissectrices des angles du triangle, et A_1', B_1', C_1' les pieds des trois bissectrices des angles supplémentaires. On sait que A_1', B_1', C_1' sont en ligne droite, ainsi que A_1', B_1, C_1. Cela posé si nous appelons p le demi-périmètre du triangle, R le rayon du cercle circonscrit, r le rayon du cercle inscrit, d la distance des centres de ces deux cercles, démontrer que les distances P, P′ du centre de gravité de ABC aux droites $A_1'B_1'C_1'$ et $A_1'B_1C_1$ sont respectivement

$$P=\frac{p^2+r(4R+r)}{6d},\quad P'=\frac{(p-a)^2-r_a(4R-r_a)}{6d_a},$$

r_a étant le rayon du cercle exinscrit tangent au côté BC et d_a la distance du centre de ce cercle au centre du cercle circonscrit 26

14573. (Professor E. J. Nanson.)—Prove that

$$\Sigma\sin^3A\sin(B-C)=-\sin(A+B+C)\,\Pi\sin(B-C) \quad \ldots\ldots 37$$

14574. (Professor A. Droz-Farny.)—La bissectrice intérieure de l'angle A du triangle ABC coupe en D et E le côté BC et la circonférence circonscrite au triangle. Soit I le centre du cercle inscrit. Calculer le rapport AI/DE. Que devient ce rapport lorsque les côtés du triangle sont en progression arithmétique, BC étant le côté moyen ? 102

14576. (Professor Cochez.)—Courbe $(x^2-y^2)xy=1$ 64

14577. (Professor Langhorne Orchard, M.A., B.Sc.)—Sum to n terms the series $1+5+3+4+3\frac{1}{2}+3\frac{3}{4}+3\frac{5}{8}+\ldots$ 48

14579. (Professor N. Bhattacharyya.)—The axes of a given ellipse E are $2a$, $2b$; and a similar ellipse E_1 is inscribed within the evolute of E; another similar ellipse E_2 is inscribed within the evolute of E_1, and so on. If S_n denote the sum of the areas of the ellipses E, E_1, E_2, ..., E_{n-1}, prove that

$$S_{2n}=\left\{1+\left(\frac{a^2-b^2}{a^2+b^2}\right)^{2n}\right\}S_n \quad \ldots\ldots 31$$

14583. (J. Macleod, M.A.)—EBK, HCL, two unequal circles external to each other, are cut by the line of centres AD in B, C. AM = diameter of EBK, and DN = diameter of HCL; M, N being on line of centres. ME is tangent to EBK, and NH to HCL. EB and NH meet in F, and ME and HC in G. Prove that $FG=\frac{1}{2}EH$ 44

14584. (R. F. Davis, M.A.)—ABC is a given triangle; O circumcentre, R circumradius. If along AO AP be set off $=R\cos^2\frac{1}{2}A$, the circle centre P, radius PA touches the incircle of ABC (as well, of course, as the circumcircle). Hence, if A be a fixed point on a given

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circle, and PQ a variable chord of constant length, the incircle of the triangle APQ touches a fixed circle. (Due to Mr. C. E. McVicker, M.A.) 112

14585. (J. H. Taylor, M.A.)—If three circles, which touch a fixed straight line at the focus S of a parabola, touch the parabola in P, Q, R, and cut in points p, q, r a circle which, centred on the axis, passes through S and touches the parabola, the circles $p\mathrm{SO}'$, $q\mathrm{SO}'$, $r\mathrm{SO}'$ (where O′ is the reflection of the focus in the directrix) intersect at 60° 93

14586. (R. C. Archibald, M.A., Ph.D.) — Any circle through the cusp and centre of the base of a cardioid cuts the curve again in two points P_1, P_2, and its double tangent in P_3, P_4. Show that, (1) the line P_1P_2 produced is tangent to the cardioid; (2) the lines P_1P_3, P_2P_4 (which do not meet unless produced) are tangent to the cardioid at P_1, P_2 respectively. Conversely, if any tangent to a cardioid meet the curve again in P_1, P_2, and if the tangents to the curve at P_1, P_2 meet the double tangent in P_3, P_4, the points P_1, P_2, P_3, P_4 lie on a circle through the cusp and centre of the base of the cardioid 127

14588. (R. P. Paranjpye, B.A.) — Prove by elementary considerations, and without the use of continued fractions, that the most general positive integral solution of the equation $x^2-2y^2=z^2$ is

$$x = k(\lambda^2+2\mu^2), \quad y = 2k\lambda\mu, \quad z = \pm k(\lambda^2-2\mu^2),$$

where λ and μ are prime to each other, and the ambiguous sign is chosen to make z positive 119

14589. (Rev. J. Cullen.) — If α, β, γ are vectors from a point P to the vertices of a given triangle ABC; also p, q, r are the trilinear coordinates of P, and p_1, q_1, r_1 those of another point Q, show that the vector PQ is $\Sigma ap_1\alpha/\Sigma ap_1$, with the condition $\Sigma ap\alpha = 0$. [*E.g.*, $\Sigma a^2\alpha/\Sigma a^2$ with $\Sigma a = 0$ is the vector from the centroid to the symmedian point.] Apply the above result to prove Quest. 14532 44

14590. (G. D. Wilson, B.A.) — If the conic represented by $lx^2+my^2+nz^2=0$ (areal coordinates) be a parabola, show that the equation of its axis is

$$\frac{\cos A}{al}(my-nz)+\frac{\cos B}{bm}(nz-lx)+\frac{\cos C}{cn}(lx-my)=0 \ldots\ldots 55$$

14592. (Lt.-Col. Allan Cunningham, R.E.)—Solve in integers in a general form the equation $N_1.N_3 = N_2.N_4$, where $N_r = x_r^4+4y_r^4$, and give numerical examples 43

14593. (Robert W. D. Christie.)—Find *all* the primitive roots of any prime of the form $12l+1$ from a *single* equation

$$ax^{2k}\pm bx^k-c \equiv 0 \bmod 12l+1,$$

and extend the same, if possible, to all primes.
E.g., $x^2\pm 5x-1 \equiv 0 \pmod{13}$ 116

14594. (V. Daniel.) — Points P and Q are taken on the sides CA, CB of a triangle ABC, so that PC.QC = PA.QB. Show that the envelope of the line PQ is the parabola touching CA and CB in A and B, and for which semi-latus rectum $= \Delta^2/\mathrm{CV}^3$, CV being the median through C 27

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14595. (R. Knowles.)—If d denote the distance between the centres of the nine-point and circum-circles of a triangle, prove that

$$4d^2 = R^2(1-8\cos A\cos B\cos C) \quad \ldots\ldots\ldots\ldots\ 49$$

14597. (Professor Neuberg.)—Trouver les relations entre les abscisses de quatre points pris sur l'hyperbole équilatère $xy = 1$, et tels que les tangentes en ces points touchent une même circonférence 84

14601. (Professor K. J. Sanjána.)—An object is placed in front of a combination of three coaxial convex lenses, whose focal lengths are f_1, f_2, f_3 respectively, and the successive distances between them a, b. If the final image is in the same position as the object, prove that

$$(f_1-a)(f_3-b)+f_2(f_1-a+f_3-b) = 0 \quad \ldots\ldots\ldots\ldots\ (1);$$

$$2f_1f_2(a+b)+2f_1f_3a-f_1b(2a+b)-f_2(a+b)^2-f_3a^2+ab(a+b) = 0 \ldots\ (2);$$

and

$$(a+b)^2f_2+a^2f_3-ab(a+b) = 0 \quad \ldots\ldots\ldots\ldots\ (3).$$

If a, b be given, the focal lengths are $a(a+b)/2b$, $ab/2(a+b)$, $b(a+b)/2a$ respectively, and the final image is equal to the object in magnitude. [A particular case is the system formed by three convex lenses of focal lengths $\frac{3}{2}$, $\frac{1}{6}$, $\frac{3}{2}$, at intervals 1, $\frac{1}{2}$; see Wolstenholme, No. 2781] 91

14602. (Professor N. Bhattacharyya.) — A light string *just fits* round an ellipse (eccentricity $= e$), and is under the action of a central force towards a focus, varying inversely as the distance. If T and T′ be the tensions at the extremities of any focal chord, show that $\epsilon^{-(T/\mu)}+\epsilon^{-(T'/\mu)}$ is constant. If N be the normal pressure on the ellipse at any point whose distance from the centre of force is r, prove that

$$N = \frac{b\mu}{\sqrt{(2ar-r^2)}}\left\{\frac{1}{r}+\frac{a}{(2ar-r^2)}\log\frac{r}{a(1-e)}\right\} \quad \ldots\ldots\ldots\ 108$$

14603. (J. A. Third, M.A., D.Sc.)—U is a straight line in the plane of a triangle ABC, Δ_1 is the conic with respect to which U is the polar of the centroid, and ABC self-polar. Prove the following statements:—(i.) If a variable in-conic, touching BC, CA, AB at X, Y, Z, has its centre on U, AX, BY, CZ cointersect on a circum-conic Δ_2. (ii.) If a variable circum-conic has its centre on U, and if LMN be the triangle formed by the tangents to it at A, B, C, AL, BM, CN cointersect on a conic Δ_3, passing through the mid-points of the sides of ABC. (iii.) Δ_1, Δ_2, Δ_3 are similar and similarly situated, and intersect on U. (iv.) The centre of Δ_3 bisects the distance between the centres of Δ_1 and Δ_2; the line of centres passes through the centroid; the centroid and the centre of Δ_1 are the centres of similitude of Δ_2 and Δ_3. (v.) Δ_1, Δ_2, Δ_3 are parabolas, hyperbolas, or ellipses, according as U touches, cuts, or does not cut in real points the maximum inscribed ellipse of ABC; they are circles when U is the line $\Sigma a\cos A = 0$, and equilateral hyperbolas when U passes through the symmedian point; generally, their eccentricity is constant, as U envelops a conic having double contact with the maximum inscribed ellipse on the line $\Sigma a\cos A = 0$. (vi.) If a variable circum-conic pass through the centre of Δ_1, its centre-locus is Δ_3, and if L′M′N′ be the triangle formed by the tangents to it at A, B, C, AL′, BM′, CN′ cointersect on U 79

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14604. (Professor Umes Chandra Ghosh.) — If a conic circumscribing a quadrilateral ABCD be inscribed in a quadrilateral *abcd*, *ab* touching the conic at D, *bc* at A, *cd* at B, and *da* at C, prove that the following quartets of lines are concurrent and harmonically conjugate:— (1) DB, AC, *bd*, *ac* (see Salmon's *Conics*, p. 242); (2) DA, BC, *ac*, *ef*; (3) AB, DC, *bd*, *ef*; and that the three diagonals of either quadrilateral form a self-conjugate triangle with respect to the conic 104

14605. (D. Biddle.)—In a tetrahedron, the trilinear coordinates of the point of contact of the inscribed sphere with any face, taken in reference to the edges of that face, are in direct proportion to the combined perpendiculars to the respective edges, from the opposite vertex and its projection on the plane of the face, the ratio being that between the area of the particular face and the entire surface of the tetrahedron 132

14606. (R. Tucker, M.A.)—ABC, A′B′C′ are the in- and circum-triangles of the same circle, and their sides are parallel. Prove

$$AB = 4\Pi (\sin \tfrac{1}{2}A)\, A'B',$$

and that A′A, B′B, C′C meet in

$$(s-a)\,\alpha/a = (s-b)\,\beta/b = (s-c)\,\gamma/c \quad \ldots\ldots\ldots\ldots\ 59$$

14607. (Prebendary Whitworth, M.A.)—"Find two unequal integers such that the sum of their reciprocals shall be equal to the reciprocal of a given integer N."—Show that there are

$$\tfrac{1}{2}(2\alpha+1)(2\beta+1)(2\gamma+1)\ldots - \tfrac{1}{2}$$

solutions of this question, where the prime factors of N are given by

$$N = a^{\alpha} b^{\beta} c^{\gamma} \ldots .$$

For example, if N = 12, there are seven solutions, viz.,

$$\tfrac{1}{12} = \tfrac{1}{13} + \tfrac{1}{156} = \tfrac{1}{14} + \tfrac{1}{84} = \tfrac{1}{15} + \tfrac{1}{60} = \tfrac{1}{16} + \tfrac{1}{48} = \tfrac{1}{18} + \tfrac{1}{36} = \tfrac{1}{20} + \tfrac{1}{30} = \tfrac{1}{21} + \tfrac{1}{28} \quad \ldots\ldots\ 85$$

14610. (R. F. Davis, M.A.)—Prove that the length of the chord of the circumcircle to a triangle ABC, drawn so as to bisect the sides AB, AC, is $2R\{1-\cos^2 B \cos^2 C\}^{\frac{1}{2}}$ 46

14613. (Rev. J. Cullen.)—Two points P and Q lie on the circumcircle of a given triangle ABC, and subtend a constant angle at O, the centre. Prove that the locus of the intersection of the Simson-lines of P and Q is a hypotrochoid 107

14614. (Lt.-Col. Allan Cunningham, R.E.)—Factorize completely $(17927+1)$ 90

14615. (W. R. Hughes.)—The four points determined by the intersection of the tangents and common chords of the circles of curvature at two points P, Q on an ellipse lie on a circle coaxal with the circles of curvature 97

14616. (R. W. D. Christie.)—Prove for primes of the form 4M−1 the modular equation $a^2 + b = 0 \bmod p$,

where a and b are any primitive roots; *e.g.*, prime 11, roots 2, 6, 7, 8, we have $2^2+7 = 6^2+8 = 7^2+6 = 8^2+2 = 0 \bmod 11$ 66

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14617. (V. Daniel.)—The locus of the middle point of the intercept of a variable tangent by tangents from a fixed point to a parabola is the tangent parallel to the polar of the fixed point 42

14621. (H. W. Curjel, M.A.)—If quadrilaterals can be at the same time inscribed in a conic C_1 and circumscribed to a conic C_2, prove that triangles can be at the same time inscribed in C_1 and circumscribed to the conic C_3, which is the polar reciprocal of C_1 with respect to C_2; and, if P is the common vertex of such a triangle and quadrilateral, the vertex of the quadrilateral opposite to P and the side of the triangle opposite to P are pole and polar with respect to C_2 105

14622. (Rev. T. Roach, M.A.)—If $T_1T_2T_3$ is a triangle described about an ellipse, then the angle subtended by T_2, T_3 at $S = \frac{1}{2}$ complement of that subtended by P, Q, the points where the tangents from T_1 touch. ["Complement" here denotes $2\pi - PSQ$] 41

14623. (Professor G. B. Mathews, F.R.S.)—Prove that

$$\Sigma E\left(\tfrac{2}{3}n\right) x^n = (x^3 + x^5)/\{(1-x)(1-x^5)\},$$

and show how to find a similar expression for $\Sigma E\{an/(a+b)\}\, x^n$. [As usual, $E(x)$ is the integer defined by $E(x) \leqslant x < E(x)+1$] 42

14624. (Professor E. Lemoine.) — Soient en coordonnées normales $a(m-n)$, $b(n-l)$, $c(l-m)$ les coordonnées d'un point M de la droite de Lemoine; soit $A_1B_1C_1$ le triangle formé par les pieds des perpendiculaires abaissées de M respectivement sur BC, CA, AB; soit un triangle A'B'C' homothétique à $A_1B_1C_1$. (1) Si AA', BB', CC' concourent en O, le lieu de O est une hyperbole équilatère circonscrite à ABC; si AB', BC', CA' concourent en O', ou AC', BA', CB' concourent en O'', les lieux de O' et de O'' sont aussi des hyperboles équilatères circonscrites à ABC. (2) Donner les équations des trois courbes 73

14627. (Professor S. Sircom.)—Trace the curve

$$(x-2)^2\{(x-2)y^2-(6-x)x^2\}^2 = 64(3-x)x^2y^2,$$

with special reference to its shape in the neighbourhood of the origin 65

14633. (D. Biddle.)—A certain group of cubic equations of the form $x^3+qx+r=0$, where q and r are both minus quantities, is capable of transformation into $y^3-2y-2=0$, by taking $y=mx$. Prove that, with the same values of q and r respectively, $z^3+z^2+\frac{1}{2}(q\pm r)=0$ and $z^3-z^2-\frac{1}{2}(q\mp r)=0$ have a root in common. Also, from the foregoing law, deduce a method of finding either of the two equations in z, having given the fellow to it only 77

14635. (R. F. Davis, M.A.)—Prove that (1) the trilinear coordinates of the foci of the ellipse touching the sides of the triangle of reference ABC at the feet of the perpendiculars are given by the equations $\beta/\gamma+\gamma/\beta = t\cos A$, $\gamma/\alpha+\alpha/\gamma = t\cos B$, $\alpha/\beta+\beta/\alpha = t\cos C$, where $t^2\cos A\cos B\cos C - t + 2 = 0$; (2) these foci all lie on the conics $a^2\cot A+\beta\gamma\operatorname{cosec} A = \ldots = \ldots$; (3) F is a focus when, HL being drawn perpendicular upon AG (Quest. 13878), KF is taken a mean proportional between KA and KL, and bisects the angle between them. [G = centroid, H = orthocentre, K = symmedian point] 131

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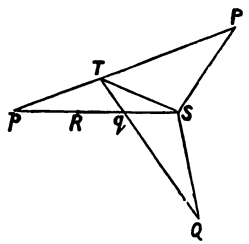
14638. (Rev. T. Roach, M.A.)—Let TP be a tangent to a parabola meeting the axis in p. Prove the following construction for the second tangent:—Centre S, radius ST, describe circle cutting Sp in R, and let

$$Sp : SR = SR : Sq ;$$

then Tq is the second tangent 84



14644. (V. Daniel.) — If a chord of the parabola $y^2 = 4ax$ and the tangents at its extremities form a triangle of constant area $\frac{1}{2}\lambda^3 a^2$, the pole of the chord lies on the parabola $y^2 = 4ax + \lambda^2 a^2$ 136

14645. (W. R. Hughes.)—PP′, PM are diameters of an ellipse and a circle, osculating at P, and MP′ meets the ellipse again in Q. Prove that the tangents at M and Q and the common chord PR meet in a point. 121

14647. (Salutation.) — ABC being any plane triangle, produce CA to A′, AB to B′, and BC to C′, so that A′B′C′ may be similar to another given triangle, and m times as large as ABC 57

14648. (Prebendary Whitworth, M.A.) — If a straight line be divided at random into four parts, the expectation of the sum of the squares on the greatest and least parts is to the expectation of the sum of the squares on the other two parts as 53 to 19. (Solution by Algebra) 71

14649. (H. W. Curjel, M.A.) — If in hyperspace of n dimensions we call the hypersphere of highest possible dimensions simply a sphere, and a space of $(n-1)$ dimensions simply a plane, and the figure corresponding to a triangle in space of 2 dimensions and a tetrahedron in space of 3 dimensions a $(n+1)$-hedroid, prove that the following theorem is true if n is even and not true if n is odd:—In space of n dimensions the $n+2$ spheres circumscribing the $n+2$ $(n+1)$-hedroids determined by $n+2$ planes meet in a point 76

14653. (Professor E. J. Nanson.)—If

$$a = (x^2-y)/(1-xy), \quad b = (y^2-x)/(1-xy),$$

prove that $(a^2-b)/(1-ab) = x$, $(b^2-a)/(1-ab) = y$ 57

14654. (Professor Neuberg.) — On donne dans un même plan deux droites Ox, Oy et un point A. Trouver un point M qui soit pour les poids donnés α, β, γ le centre de gravité du point A et des projections B, C de M sur Ox, Oy. (Solution géométrique) 31

14656. (Professor K. J. Sanjána, M.A.)—(1) If a, b, c be positive quantities, every two of which are together greater than the third, prove that $\Sigma a^3 - \Sigma a^2 b + 3abc$ is never negative. Hence show that in a plane triangle the inradius is never greater than half the circumradius. (2) With the ordinary notation for a plane triangle, prove that

$$\Sigma a^2 b - \Sigma a^3 = 8S(R+r) \text{ ..... } 85$$

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14657. (Professor Langhorne Orchard, M.A., B.Sc.)—Show that

$$3\left\{1^5+2^5+3^5+\ldots+n^5-(1+2+3+\ldots+n)^3\right\} = (1+2+3+\ldots+n)^3-(1+2+3+\ldots+n)^2,$$

n being a positive integer 108

14662. (R. C. Archibald, M.A.)—In the xy-plane, passing through the origin S, and with its centre on the x-axis, is a circle of radius a. On this circle is rolled an equal circle whose plane is always perpendicular to the xy-plane. The point P of this rolling circle, which was at the beginning of the motion in coincidence with S, traces out a curve C. Show that (1) the curve C lies both on a sphere of radius $a\sqrt{2}$ and on the cone $x^2+y^2=z^2$; (2) the orthogonal projection of C on the xy-plane is the cardioid whose axial circle is the given fixed circle; (3) the projection of C on the yz-plane is the cuspidal quartic called by the French a *toupie* or *piriforme*; (4) the projection of C on the xz-plane is a parabola 109

14667. (Lt.-Col. Allan Cunningham, R.E.) — Factorize *completely* (into prime factors) $N = 7^{35}+1$ 111

14668. (Robert W. D. Christie.) — Prove the following modular equations for any prime:—

$$r^{2m-n}+r_1{}^n \equiv 0 \bmod p = 0 \bmod 4m+1,$$

$$r^{2m-n-1}+r_1{}^n = 0 \bmod p = 0 \bmod 4m-1,$$

r and r_1 being associated primitive roots, and n any integer whatever 70

14669. (V. Daniel.)—Triangles are inscribed one to another so that the sides of one are divided, in the same sense, by the vertices of the next in a constant ratio $k/(1-k)$. Show that the sides of the nth inscribed triangle, in terms of a, b, c (the sides of the outermost), are given by

$$-3a_n{}^2 = 4\lambda^n\,(a^2\cos\theta_1\cos\theta_2+b^2\cos\theta_2\cos\theta_3+c^2\cos\theta_3\cos\theta_1),$$

and two similar expressions with a, b, c, changed in cyclic order, where $\cos\theta_1$, $\cos\theta_2$, $\cos\theta_3$ are the roots of the trigonometrical cubic

$$4\cos^3\theta-3\cos\theta+\sin 3n\phi = 0,$$

and $$\phi = -\tfrac{1}{6}\pi+\sin^{-1}\tfrac{1}{2}\lambda^{-\frac{1}{2}} = \tfrac{1}{3}\cos^{-1}\left(\tfrac{3}{2}\lambda^{-\frac{1}{2}}-\tfrac{1}{2}\lambda^{-\frac{3}{2}}\right),$$

and $$\lambda = 1-3k+3k^2$$ 85

14672. (Salutation.) — Within the square ABCD the semicircle AIB is described; from A a straight line is drawn, cutting the semicircle in P; through P another line is drawn, perpendicular to AB, cutting CD in M; and finally MN is drawn perpendicular to AP produced. Find the locus of N 106

14673. (Rev. T. Roach, M.A.)—A countryman, unversed in numbers, picks a bulrush, which he breaks into three pieces, and gives the longest to his eldest nephew Dick, the second to Sam, the next in age, and the third to Bob, the youngest. At the same time he promises each a ball of toffy as high as the length of his bulrush. Find the value of the expectation of each nephew, given that the whole bulrush was 24 inches long and that toffy costs $1\frac{1}{4}$d. a cubic inch, and taking the value of π as $3\frac{1}{7}$ 133

14674. (H. J. Woodall, A.R.C.S.)—Solve $x^{23} \equiv 1 \pmod{1933}$ ... 69

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14675. (The late Professor Wolstenholme, M.A., Sc.D.) — In the tetrahedron OABC the sum of the faces ABC, OBC is equal to the sum of the faces OCA, OAB. Prove that

$$(a-x)/\sin\tfrac{1}{2}(X-A) = (b-y)/\sin\tfrac{1}{2}(B-Y) = (c-z)/\sin\tfrac{1}{2}(C-Z)$$
$$= \text{some positive quantity.}$$

[The lengths OA, OB, OC are denoted by a, b, c; and the lengths BC, CA, AB by x, y, z; and the dihedral angles opposite these edges by A, B, C, X, Y, Z respectively] 64

14677. (Professor Jan de Vries.) — The sides of three complete $(2n+1)$-laterals are touched by a conic. Prove that their vertices are joined by a curve of degree $3n$ 115

14680. (Professor Sanjána, M.A.)—Prove that the locus of points from which tangents drawn to the ellipse $x^2/a^2 + y^2/b^2 - 1 \equiv S = 0$ form with their chords of contact triangles whose orthocentres lie on S is the sextic

$$x^4y^2a^2(a^2-b^2)^2 + x^2y^4b^2(a^2-b^2)^2 - 2x^2y^2(a^8 - a^4b^4 + b^8)$$
$$- x^4a^2b^6 - y^4b^2a^6 + x^2b^6(a^2+b^2)^2 + y^2a^6(a^2+b^2)^2 = 0.$$

Show that this sextic breaks up into the conic $a^2x^2 + b^2y^2 = (a^2+b^2)^2$ and the quartic $b^6x^2 + a^6y^2 = x^2y^2(a^2-b^2)^2$; and explain the result. In the parabola $y^2 = 4ax$ the locus is the quartic

$$(x+2a)(xy^2 + 2ay^2 + 4a^3) = 0.$$

Explain the appearance of the two factors 114

14684. (D. Biddle.)—If a tetrahedron have its opposite edges equal, the point of contact of the inscribed sphere with any face is at the circumcentre of the face 133

14685. (R. Tucker, M.A.)—Find the H.C.F. of

$$\begin{vmatrix} \cos(C-\omega)\cos(A-\omega) & -\cos\omega\cos(A-\omega) & \cos^2\omega \\ \cos^2\omega & \cos(A-\omega)\cos(B-\omega) & -\cos\omega\cos(B-\omega) \\ -\cos\omega\cos(C-\omega) & \cos^2\omega & \cos(B-\omega)\cos(C-\omega) \end{vmatrix},$$

and

$$\begin{vmatrix} c\cos(A-\omega) & -a\cos\omega\cos(A-\omega) & a\cos^2\omega \\ a & b\cos(A-\omega) & -b\cos\omega \\ b\cos(C-\omega) & c\cos^2\omega & c\cos(B-\omega)\cos(C-\omega) \end{vmatrix}$$

..... 110

14704. (Professor G. B. Mathews, F.R.S.)—If x, y independently assume the values 0, ±1, ±2, ±3, &c., prove that

$$\Sigma e^{-\pi(x^2+2y^2)} = \frac{\sqrt{(4+2\sqrt{2})}}{\pi}\int_0^1 \frac{dx}{\sqrt{(1-x^4)}} \quad \text{..... 58}$$

14705. (Professor Neuberg.) — Etant donné un triangle ABC, appelons a', b', c' les distances B'C', C'A', A'B' des projections A', B', C' d'un point M sur les côtés BC, CA, AB. Trouver le lieu des points M lorsque a', b', c' vérifient la relation donnée $ka'^2 + lb'^2 + nc'^2 = p$. Trouver également le lieu décrit par le centre de gravité du triangle A'B'C'... 75

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14713. (Prince C. de Polignac.)—If t_1, u_1 are the smallest integer solutions of the Pellian equation $t^2 - Du^2 = 1$ (D a positive integer, not a square), t_n, u_n any other two solutions, show that there exists a linear substitution $x_1 = (Q_1x + S_1)/(P_1x + R_1)$ such that, if we write its nth power as $x_n = (Q_nx + S_n)/(P_nx + R_n)$, Q_n and P_n/u_1 will give the solutions t_n, u_n of the above equation 67

14724. (R. F. Muirhead.) — Through a given point E draw a line QEP meeting AB produced in Q and BC produced in P so that BQ = CP, A, B, C being given points 101

14750. (R. P. Paranjpye, B.A.)—Find the limit, if there be any, of

$$4^{-4^{-4^{\text{ad inf.}}}}$$

..... 42

14797. (Professor N. Bhattacharyya.)—A system of co-axial circles intersect in real points S, S′; two lines $SP_1P_2 \ldots$, $SQ_1Q_2 \ldots$ are drawn to intersect them in points P_1, P_2, P_3, ...; Q_1, Q_2, Q_3, ... Show that the envelope of the chords P_1Q_1, P_2Q_2, ... is a parabola whose focus is the point S′. If through any point T in the line SS′ a line be drawn to intersect the circles in A_1, A_2; B_1, B_2; C_1, C_2; &c., and if O be any point in the plane of the circles, show that the circles circumscribing the triangles OA_1A_2, OB_1B_2, ..., OSS′ will form a coaxial system 132

14798. (Professor Umes Chandra Ghosh.) — Find the envelope of Simson's lines of a given triangle with respect to points on its circumference 107

CORRIGENDA.

Vol. LXXIV., p. 38, l. 6 from foot, *for* "r^4" *read* "ν^4."

p. 62, ll. 10 and 11, *add* "dx" to first integrals.

p. 147, l. 2 from foot, *for* "suffixes supply" *read* "suffixes apply."

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MATHEMATICS

FROM

THE EDUCATIONAL TIMES,

WITH ADDITIONAL PAPERS AND SOLUTIONS.

14569. (JOHN C. MALET, F.R.S.)—Being given that the roots of an algebraic equation of the twelfth degree, α_1, α_2, α_3, α_4, β_1, β_2, β_3, β_4, γ_1, γ_2, γ_3, γ_4, are connected by the relations

$$\alpha_1+\alpha_2+\alpha_3+\alpha_4 = \beta_1+\beta_2+\beta_3+\beta_4 = \gamma_1+\gamma_2+\gamma_3+\gamma_4,$$
$$\alpha_1\alpha_2\alpha_3\alpha_4 = \beta_1\beta_2\beta_3\beta_4 = \gamma_1\gamma_2\gamma_3\gamma_4,$$

solve the equation.

Solution by the PROPOSER; *and* H. W. CURJEL, M.A.

Let
$$u_1 = \Sigma\alpha_1\alpha_2, \quad u_2 = \Sigma\beta_1\beta_2, \quad u_3 = \Sigma\gamma_1\gamma_2,$$
$$v_1 = \Sigma\alpha_1\alpha_2\alpha_3, \quad v_2 = \Sigma\beta_1\beta_2\beta_3, \quad v_3 = \Sigma\gamma_1\gamma_2\gamma_3.$$

Then, from the given conditions, we easily find

$$\Sigma u_1 + \tfrac{1}{8}p_1^2 = p_2, \quad \Sigma u_1u_2 + \tfrac{3}{8}p_1\Sigma v_1 + \tfrac{1}{16}p_1^2\Sigma u_1 + 3p_{12}^{\frac{1}{2}} = p_4,$$
$$\Sigma u_1(v_2+v_3) + \tfrac{1}{8}p_1\Sigma u_1u_2 + \tfrac{1}{16}p_1^2\Sigma v_1 + 2p_1p_{12}^{\frac{1}{2}} = p_5,$$
$$u_1u_2u_3 + \Sigma v_1v_2 + 2p_{12}^{\frac{1}{2}}\Sigma u_1 + \tfrac{1}{16}p_1\Sigma u_1(v_2+v_3) + \tfrac{1}{8}(p_1^2p_{12}^{\frac{1}{2}}) = p_6,$$
$$v_1v_2v_3 + p_{12}^{\frac{1}{2}}\Sigma u_1(v_2+v_3) + p_1p_{12}^{\frac{1}{2}} = p_9,$$
$$p_{12}^{\frac{1}{2}}\Sigma v_1v_2 + p_{12}^{\frac{1}{2}}\Sigma u_1 = p_{10}, \quad p_{12}^{\frac{1}{2}}\Sigma v_1 = p_{11},$$

where p_1, p_2, &c. are the coefficients of the given equation. Thus we have seven simple equations for the determination of

$$\Sigma u_1, \quad \Sigma u_1u_2, \quad u_1u_2u_3, \quad \Sigma v_1, \quad \Sigma v_1v_2, \quad v_1v_2v_3, \quad \Sigma u_1(v_2+v_3).$$

Hence u_1, u_2, u_3 are determinable as the roots of one cubic equation, and v_1, v_2, v_3 as the roots of another, and the roots of the equation of the twelfth degree are the roots of the quartics

$$x^4-(\tfrac{1}{8}p_1)x^3+u_1x^2-v_1x+p_{12}^{\frac{1}{2}} = 0, \quad x^4-(\tfrac{1}{8}p_1)x^3+u_2x^2-v_2x+p_{12}^{\frac{1}{2}} = 0,$$
$$x^4-(\tfrac{1}{8}p_1)x^3+u_3x^2-v_3x+p_{12}^{\frac{1}{2}} = 0.$$

Hence, &c.

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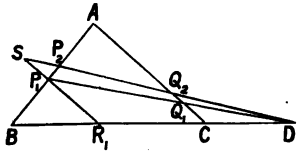
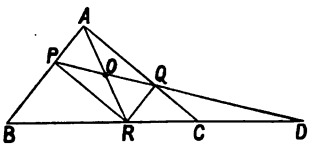
6721. (J. Young, B.A.)—Through a point in the base of a triangle produced, draw a straight line cutting the sides so that the rectangle contained by the segment of one side towards the base and the segment of the other towards the vertex of the triangle shall be a maximum.

Solution by James Blaikie, M.A.; *and* J. Heber Steward.

Let ABC be the triangle, and let DQ_1P_1, DQ_2P_2 be two consecutive positions of the secant on each side of the position for which $AP.CQ$ is a maximum, such that $AP_1CQ_1 = AP_2.CQ_2$. Subtracting $AP_2.CQ_1$ from both sides of this equation, we have $P_1P_2.CQ_1 = AP_2.Q_1Q_2$ or $AP_2/CQ_1 = P_1P_2/Q_1Q_2$. Through P_1 draw SP_1R_1 parallel to AC. Then $AP_2/CQ_1 = P_1P_2/P_1S.P_1S/Q_1Q_2 = AP_2/AQ_2.P_1R_1/Q_1C$. Therefore $AQ_2 = P_1R_1$, and, in the limit when P_1 coincides with P_2, APRQ is a parallelogram whose diagonal passes through D. Therefore $DB/DR = PB/QR = PR/QC = DR/DC$, or $DR^2 = DB.DC$. Thus R is known. Join AR,





bisect it in O, and join DO, cutting AB, AC in P, Q. The rectangle AP.CQ is a maximum. A similar reasoning gives us the same construction in order that AQ.BP may be a maximum.

If DR′ be taken on BD produced, we obtain another maximum, the secant P′DQ′ being parallel to AR and cutting BA and AC produced.

It may be observed that $AP : PB = CQ : QA = CR : RB = \sqrt{(DC)} : \sqrt{(DB)}$, and that the triangles PBR, QRC are proportional to the rectangles AQ.BP, AP.CQ.

14572. (Professor E. Lemoine.)—Soit ABC un triangle. Appelons A_1, B_1, C_1 les pieds sur BC, CA, AB des trois bissectrices des angles du triangle, et A_1', B_1', C_1' les pieds des trois bissectrices des angles supplémentaires. On sait que A_1', B_1', C_1' sont en ligne droite, ainsi que A_1', B_1, C_1. Cela posé si nous appelons p le demi-périmètre du triangle, R le rayon du cercle circonscrit, r le rayon du cercle inscrit, d la distance des centres de ces deux cercles, démontrer que les distances P, P′ du centre de gravité de ABC aux droites $A_1'B_1'C_1'$ et $A_1'B_1C_1$ sont respectivement

$$P = \frac{p^2 + r(4R + r)}{6d}, \quad P' = \frac{(p-a)^2 - r_a(4R - r_a)}{6d_a},$$

r_a étant le rayon du cercle exinscrit tangent au côté BC et d_a la distance du centre de ce cercle au centre du cercle circonscrit.

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Solution by R. F. DAVIS, M.A.; *and* Professor SANJÁNA, M.A.

If O be the circumcentre and I the incentre, it may be shown that OI is perpendicular to $A_1'B_1'C_1'$ $(a+\beta+\gamma=0)$. Draw AL, BM, CN, GH perpendiculars to the same line; OF, IP to AB. Then

$$FP : OI = AL : AC_1' \quad \text{and} \quad AL \cdot d = \{bc/(a-b)\}\tfrac{1}{2}(a-b) = \tfrac{1}{2}bc.$$

Similarly $\quad BM \cdot d = \tfrac{1}{2}ca, \quad CN \cdot d = \tfrac{1}{2}ab.$

Hence $\quad GH \cdot 6d = \Sigma bc = p^2 + r(4R+r).$

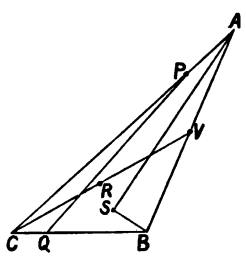
(*Reprint*, Vol. LXVI., p. 111, Quest. 13298.)

14594. (V. DANIEL.)—Points P and Q are taken on the sides CA, CB of a triangle ABC, so that PC.QC = PA.QB. Show that the envelope of the line PQ is the parabola touching CA and CB in A and B, and for which semi-latus rectum $= \Delta^2/CV^3$, CV being the median through C.

Solution by the PROPOSER; PANCHAPAGESA AIYAR; *and others.*

Take CA, CB as axes. Equation to PQ involving a parameter k is

$$\frac{x}{ka} + \frac{y}{(1-k)b} = 1.$$



Differentiating,

$$-(x/a)/k^2 + y/b\,(1-k)^2 = 0;$$

therefore

$$\frac{(x/a)^{\frac{1}{2}}}{k} = \frac{(y/b)^{\frac{1}{2}}}{1-k} = (x/a)^{\frac{1}{2}} + (y/b)^{\frac{1}{2}};$$

and the envelope is

$$(x/a)^{\frac{1}{2}} + (y/b)^{\frac{1}{2}} = 1,$$

a parabola touching the axes in $x = a$ and $y = b$. CV is parallel to the axis and is bisected by the curve in R. Hence

$$a^2 = 4SB \cdot CR, \quad b^2 = 4SA \cdot CR,$$

and by the polar equation of a parabola referred to its focus

$$SA = l/2\sin^2 ACV, \quad SB = l/2\sin^2 BCV;$$

therefore $\quad ab = 4CR\,(SA \cdot SB)^{\frac{1}{2}} = 2l \cdot CR/\sin ACV \sin BCV,$

whence $\quad l = ab \sin ACV \sin BCV/CV = \Delta^2/CV^3.$

[See SALMON, *Conic Sections*, 6th edition.]

14421. (Professor IGNACIO BEYENS, M.A., Lt.-Col. du Génie).—Soient O, A, B trois points sur une droite donnée; par le point O on mène une droite quelconque OX et l'on détermine sur cette dernière le

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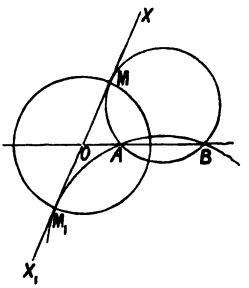
point M tel que l'angle AMB soit *maximum;* et, si l'on fait la même construction pour tous les droites qui passent par O, déterminer le lieu *géométrique des points* M.

Solution by Professor A. Droz-Farny; H. W. Curjel, M.A.; George Scott, M.A.; *and others.*

C'est une proposition bien connue que sur chaque droite OX se trouvent deux points M, les points de contact des deux circonférences tangentes à OX et passant par les points A et B.



Comme $(OM)^2 = (OM_1)^2$
$= OA.OB,$

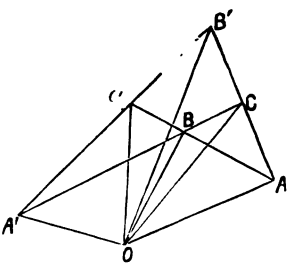
le lieu cherché est un cercle de centre O et de rayon $r = \sqrt{(OA.OB)}$.

[As the angles AMB, AM_1B clearly differ, only one can be the maximum, unless AOX be a right angle.]

14233. (Professor K. J. Sanjána.)—If three points A, B, C be referred to two axes meeting in O, the intersections of the joins of B and C, C and A, A and B, with the isogonal conjugates with regard to the given axes of OA, OB, OC, respectively, are collinear. If three straight lines BC, CA, AB be referred to two points on a line Δ, the joins of A, B, C with the isotomic conjugates with regard to the given points of the intersections of Δ with BC, CA, AB, respectively, are concurrent.

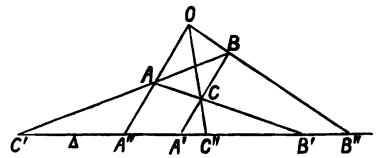
Solutions (i.) *by* H. W. Curjel, M.A.; (ii.) *by* Professor A. Droz-Farny.

(i.) (1) Let OA′, OB′ (fig. 1) be the harmonic conjugates of OA, OB with respect to the two rays OX, OY; A′, B′ being on BC, CA. Let A′B′ meet AB in C′. Then AB, BC, CA, A′C′ form a complete quadrilateral, of which A, A′; B, B′; C, C′ are pairs of opposite vertices. Therefore OA, OA′; OB, OB′; OC, OC′ are pairs of an involution of which OX, OY are the double rays. Therefore O(XY, CC′) is harmonic. Hence the first theorem of the question, if OX, OY are the bisectors of the angles between the given axes.



(2) Let BC, CA, AB meet Δ in A′B′C′, and let A″, B″ be the harmonic

conjugates of A′, B′ with respect to two points X, Y of Δ, and let A″A, BB″ meet in O. Let OC meet Δ in C″. Then ABCO is a complete quadrangle, of which AB, OC; AC, OB; BC, OA are pairs of opposite sides. Therefore A′, A″; B′, B″; C′, C″ are pairs of an involution of which X, Y are the double points. Therefore (XY, C′C″) is harmonic. Hence the second theorem, if X and Y are the bisectors of the lines joining the two given points and the point at infinity on Δ.



(ii.) Les deux propriétés énoncées sont des cas particuliers de deux théorèmes généraux énoncés par Chasles dans sa *Géométrie Supérieure*, p. 247.

(1) Si d'un même point on mène des droites aux trois sommets d'un triangle et trois autres droites formant avec ces premières trois couples en involution, ces trois droites rencontreront les côtés opposés du triangle en trois points situés en ligne droite.

(2) Si sur une transversale qui rencontre les côtés d'un triangle en trois points on prend trois autres points quelconques, formant avec ceux-là trois couples en involution, les droites menées de ces trois points aux sommets opposés du triangle concourront en un même point.

Dans le premier cas, l'isogonale de OA par rapport aux axes rencontre BC en A′; or OA et OA′, OB et OB′, OC et OC′ constituent trois couples en involution, à laquelle appartiennent les axes et dont les rayons doubles sont les bissectrices des deux angles des axes.

Dans le deuxième cas, soient x et x' les deux points; BC rencontre la transversale en a et soit a' son isotomique par rapport aux points fixes; or a et a', β et β', γ et γ' constituent une involution ponctuelle à laquelle appartiennent les points fixes et dont les points doubles sont le point milieu de xx' et commun à tous les couples et le point infini de la transversale.

14564. (R. Chartres.)—If ABC be a triangle, P and Q random points, find the mean value of the nth power of the triangle PAQ (1) when P and R are confined to the base BC, (2) when P and Q can be anywhere in the triangle. Also find the same when ABC is a sector of a circle, centre A; and deduce the former results from these.

Solution by the Proposer.

$$\Delta^n = (\Delta_1 + \Delta_2 + \Delta_3)^n, \text{ containing } \frac{(n+1)(n+2)}{2} \text{ terms};$$

therefore mean value of $(\Delta_1)^n = \dfrac{2\Delta^n}{(n+1)(n+2)}$.

Now, if P move in the element AP_1P_2, the mean value of $AP^n = \dfrac{2(AP_1)^n}{n+2}$.

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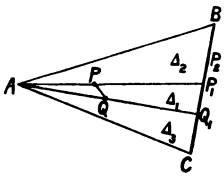
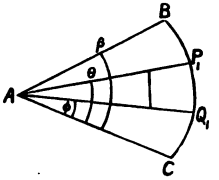
and similarly for Q; therefore mean value of $(APQ)^n = \frac{8\Delta^n}{(n+1)(n+2)^2}$.

If $n = 1$, mean value of $APQ = \frac{4}{27}\Delta$.





In the case of the sector, mean value of

$$(AP_1Q_1)^n = (\tfrac{1}{2}r^2)^n \int_0^\beta\int_0^\phi \sin^n(\theta-\phi)\,d\theta\,d\phi \div \int_0^\beta\int_0^\phi d\theta\,d\phi;$$

therefore, as above, mean value of

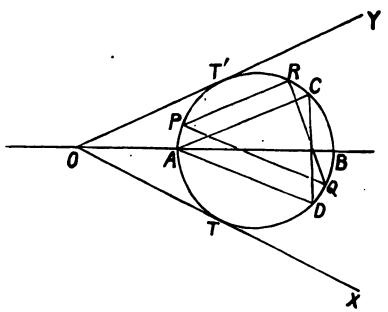
$$(APQ)^n = (\tfrac{1}{2}r^2)^n \frac{2^3}{(n+2)^2\beta^2}\int_0^\beta\int_0^\phi \sin^n(\theta-\phi)\,d\theta\,d\phi.$$

If $n = 1$, mean value of $APQ = \frac{1}{9}(r^2/\beta^2)(\beta - \sin\beta)$.

This can be arranged as $\frac{8}{9}\Delta\left(\frac{\beta-\sin\beta}{\beta^2\sin\beta}\right) = \frac{8}{9}\Delta \times \frac{1}{6} = \frac{4}{27}\Delta$

by expanding and neglecting powers of β higher than the first.

14523. (R. Tucker, M.A.)—Pairs of chords are drawn, from points on a given circumference, in fixed directions. Find the locus of the centroids of the variable triangles (see figure).



Solution by the Proposer.

Let 2ω be the constant angle between the lines, $x^2+y^2 = a^2$ the circle; then the lines PR, PQ are given by $y-k = m(x-h)$ [P, (h, k) and $m = \tan\omega$] and $y-k = -m(x-h)$; then coordinates of R, Q are easily seen to be

$$\left\{(k-mh)^2-a^2\right\}\cos^2\omega/h, \quad \left\{(mh-k)^2-m^2a^2\right\}\cos^2\omega/k,$$

and $$\left\{(k+mh)^2-a^2\right\}\cos^2\omega/h, \quad \left\{(mh+k)^2-m^2a^2\right\}\cos^2\omega/k.$$

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Hence $3\bar{x} = h(1-2\cos 2\omega)$ and $h^2+k^2 = a^2$, and $3\bar{y} = k(1+2\cos 2\omega)$; therefore locus required is the conic

$$9x^2/(1-2\cos 2\omega)^2 + 9y^2/(1+2\cos 2\omega)^2 = a^2.$$

14654. (Professor Neuberg.)—On donne dans un même plan deux droites Ox, Oy et un point A. Trouver un point M qui soit pour les poids donnés α, β, γ le centre de gravité du point A et des projections B, C de M sur Ox, Oy. (Solution géométrique.)

Solution by James Blaikie, M.A.

Draw AN perpendicular to Ox. Take OB on Ox so that

$$OB = \alpha/(\alpha+\gamma).ON;$$

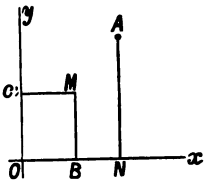
and draw BM perpendicular to Ox and

$$= \alpha/(\alpha+\beta).NA.$$

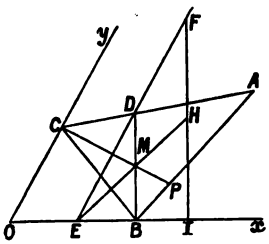
M is the point required. For

$$CM(\alpha+\beta+\gamma) = OB.\beta + ON.\alpha$$

and $$BM(\alpha+\beta+\gamma) = OC.\gamma + NA.\alpha.$$



[The Proposer sends the following:—*Solution générale.*—Je suppose l'angle xOy quelconque. Si MB coupe CA en D, on a $AD : DC = \gamma : \alpha$, $DM : MB = \beta : \alpha+\gamma$. Le lieu des points qui divisent dans le rapport $\gamma : \alpha$ les droites menées de A vers Oy est une droite EF parallèle à Oy; construisons cette droite. De même, on peut construire la droite EH, lieu des points qui divisent dans le rapport $\beta : \alpha+\gamma$ les perpendiculaires abaissées des points de EF sur Ox.



On aura une seconde droite passant par M en considérant les proportions $AP : PB = \beta : \alpha$, $PM : MC = \gamma : \alpha+\beta$.]

14579. (Professor N. Bhattacharyya.)—The axes of a given ellipse E are $2a$, $2b$; and a similar ellipse E_1 is inscribed within the evolute of E; another similar ellipse E_2 is inscribed within the evolute of E_1, and so on. If S_n denote the sum of the areas of the ellipses E, E_1, E_2, ..., E_{n-1}, prove that

$$S_{2n} = \left\{1+\left(\frac{a^2-b^2}{a^2+b^2}\right)^{2n}\right\} S_n.$$

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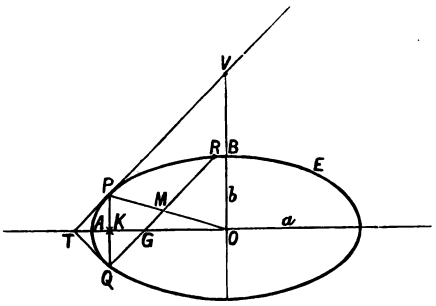
Solution by F. L. WARD, B.A.; *and* Rev. J. CULLEN.

Let TPV be the tangent to the ellipse at a point P such that $\angle VTO = 45°$. Join P to O, the centre; T and V are on the axes. Draw the chord PQ perpendicular to OT. TQ is the other tangent from T to E; and, since TP and TQ are at right angles, OT = radius of the orthocycle $= \sqrt{(a^2+b^2)} = OV$.



Let PO meet axis in K, QR be the chord through Q parallel to TV, and therefore normal at Q, and bisected in M on OP; let it meet the axis in G.

Since the chord QR and the tangent at Q are equally inclined to the axis, therefore QR is the common chord of the circle of curvature at Q and the ellipse, but, being normal at Q, it must be the diameter of the circle. Hence M is the centre of curvature, QM being tangent to the evolute at M. But, since QR is parallel to TP, therefore M is a point of contact of the similar ellipse to E inscribed in the evolute. The ratio of the axes of E_1VE is $OM : OP = OG : OT$. Now $TK = KP$ and $PK.OV = b^2$. But $OV = \sqrt{(a^2+b^2)}$; therefore $TK = KG = \dfrac{b^2}{\sqrt{(a^2+b^2)}}$;

therefore $$OG = \sqrt{(a^2+b^2)} - \frac{2b^2}{\sqrt{(a^2+b^2)}} = \frac{a^2-b^2}{\sqrt{(a^2+b^2)}};$$

therefore $$OG : OT = a^2-b^2 : a^2+b^2.$$

Hence area of E_1 : area of $E = (a^2-b^2)^2 : (a^2+b^2)^2$;

therefore $$S_n = E\left\{1 + \left(\frac{a^2-b^2}{a^2+b^2}\right)^2 + \left(\frac{a^2-b^2}{a^2+b^2}\right)^4 + \ldots + \left(\frac{a^2-b^2}{a^2+b^2}\right)^{2(n-1)}\right\}.$$

Put $$\left(\frac{a^2-b^2}{a^2+b^2}\right)^2 = k;$$

therefore $$S_n = E\frac{1-k^n}{1-k}; \qquad \text{therefore} \quad S_{2n} = \frac{1-k^{2n}}{1-k};$$

therefore $$\frac{S_{2n}}{S_n} = \frac{1-k^{2n}}{1-k^n} = (1+k^n); \quad \text{therefore} \quad S_{2n} = \left\{1 + \left(\frac{a^2-b^2}{a^2+b^2}\right)^{2n}\right\} S_n.$$

14415. (R. KNOWLES.)—ABCD is a quadrilateral inscribed in a conic so that one of its diagonals AC passes through the pole of the other. Prove that the point of intersection of AD and the tangent at B, that of AB and the tangent at D, and the pole of AC are collinear.

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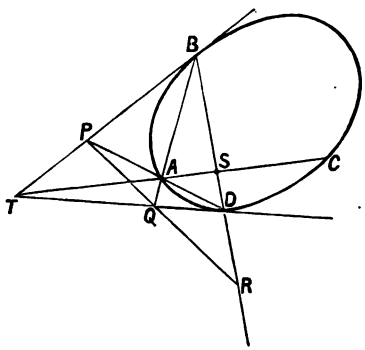
Solution by G. D. Wilson, B.A.; Professor Droz-Farny; G. G. Morrice, M.D.; *and many others.*



Let the tangents at B and D meet in T, which, by hypothesis, lies on AC. Also let AD, BT meet in P; AB, DT in Q, and BD, AC in S. Then TA, PQ, BD are the diagonals of a complete quadrilateral; so that, if PQ meet BD in R,

$$(BSDR) = -1.$$

But the pole of AC must lie on BD. Call it R′. Then it is equally true that

$$(BSDR') = -1.$$

Therefore R and R′ coincide, and P, Q, R′ are collinear.

14560. (Robert W. D. Christie.)—

A gentleman a garden had
Of a yards long and c yards broad.
A walk of equal width all round
He made that took one-fourth the ground.
Ye skilful in geometry,
Tell me how wide the walk must be.
All fractions discard and integers use,
Or time and trouble you'll certainly lose.

I. *Solution by* Rev. T. Roach, M.A.

Let x be the width of the path; then $(a-2x)(c-2x) = \frac{3}{4}ac$. Hence

$$4x = a+c \pm \sqrt{(a^2+ac+c^2)} = a+c \pm \sqrt{[(a^3-c^3)/(a-c)]}.$$

Now, by trial, a and c are equimultiples of 5 and 3. Let them $= 5y$ and $3y$; therefore $4x = 8y \pm 7y$. Hence lowest integral value of $y = 4$, and $x = 1$, $a = 20$, $c = 12$ or any equimultiples of these numbers.

II. *Solution by* Lt.-Col. Allan Cunningham, R.E.

Let x = width of path. Then $(a-2x)(c-2x) = \frac{3}{4}ac$, whence

$$x = \tfrac{1}{4}\left\{a+c \pm (a^2+ac+c^2)^{\frac{1}{2}}\right\},$$

which is to be an *integer*. This gives

$$a^2+ac+c^2 = (\tfrac{1}{2}a+c)^2 + 3(\tfrac{1}{2}a)^2 = z^2, \text{ a perfect } \square.$$

The *general* solution of this, in terms of two arbitraries (t, u), is

$$\tfrac{1}{2}a+c = t^2-3u^2, \quad \tfrac{1}{2}a = 2tu, \quad z = t^2+3u^2,$$

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whence $a = 4tu$, $c = t^2 - 2tu - 3u^2$, $x = \frac{1}{2}t(t+u)$, or $\frac{1}{2}u(t-3u)$, where t, u are *both odd* or *both even*, and $t > 3u$.

Ex.—Take $t = 5a$, $u = a$; then $a = 20a^2$, $c = 12a^2$, $x = a^2$. Area $= 240a^4$, area of path $= 60a^4 = \frac{1}{4}$ of whole.

III. *Solution by the* PROPOSER.

The problem to be solved here is evidently

$$a^2 + ac + c^2 = x^2 \quad \text{and} \quad 4x = a + c \pm z.$$

A few general solutions are as follow :—

$a = q(2p+q)$,	$c = (p+q)(p-q)$,	whence z	and x	..... (1) ;
$a = 4k(k+1)$,	$c = 4k+3$,	,,	,,	..... (2) ;
$a = 8n(k+1)$,	$c = 8n + 6n/k$,	,,	,,	..... (3) ;
$a = nk(nk+2)$,	$c = 2nk+3$,	,,	,,	..... (4) ;
$a = nk(3nk+2)$,	$c = 2nk+1$,	,,	,,	..... (5) ;
$a = n^2k^2 - 1$,	$c = 2nk+1$,	,,	,,	..... (6) ;

all the letters being arbitrary.

To secure integers for x in (1) we may take $p = q + 4r$ or $p = 4rs + 1$, $q = 4rs$, and in a similar way for the other general solutions.

6278. (Prof. HUMPIDGE, B.Sc., Ph.D.)—A material sphere being given, find the maximum number of equal spheres that may be placed round it, and touching it.

Solution by W. J. GREENSTREET, M.A.

Given a circle, the number of equal circles touching it and each other is six. Revolve the figure around the line on which three of the seven centres lie. The three circles of these centres generate three spheres; the rest generate tores, touching each other on a circle, radius $r\sqrt{3}$, each touching the original sphere and the respective external spheres. Hence we merely require the maximum number of spheres which may be inserted in the tores. Project the spheres on the common tangent plane of the two tores, and we have circles of radius r, having their centres on the circle of contact. The question thus reduces to: find the maximum number of circles, radius r, which may be placed in contact with each other and a circle, radius $r\sqrt{3}$.

Let P_1, P_2 be the centre of two such tangent circles; O the centre of the large circle. Then $\cos\theta = \frac{1}{7}$, when $P_1OP_2 = \theta$, because the sides of P_1OP_2 are 2R, $R\sqrt{3}$, $R\sqrt{3}$, *i.e.*, θ is roughly 70°. Hence there are only five circles. It follows that only five spheres can be inserted in each tore, and therefore the total number of spheres is twelve.

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14556. (R. P. Paranjpye, B.A.)—Trace the curve $(x/a)^n + (y/b)^n = 1$ when n is an infinite positive integer, distinguishing the cases when it is odd or even.

Solution by H. W. Curjel, M.A.

If n is even, x cannot be greater than a and y cannot be greater than b, when $x < a$, $y = \pm b$, and when $y < b$, $x = \pm a$; therefore curve consists of the sides of the parallelogram whose four corners are $(\pm a, \pm b)$, excluding these corners.

If n is odd, the curve clearly consists of the two sides of the parallelogram which meet in the point (a, b), and the part of the line $x/a + y/b = 0$ which is outside the parallelogram. The corners of the parallelogram must be excluded in this case also.

14533. (Rev. T. Wiggins, B.A.)—Express the numbers 131, 133, 137, 139 by four nines. Algebraical symbols and decimal points may be used, but the expression for each number must contain the four nines *and no other figure*. Also, express the number 113 by four fours, with similar conditions. (The above is part of Quest. 6487, which was reproposed in *Ed. Times*, March, 1900.)

Solution by Lt.-Col. Allan Cunningham, R.E.

$$131 = \Gamma(\sqrt{9!}) + 99 + 9; \qquad 133 = \Gamma(\sqrt{9!}) + 9 + \sqrt{9} + \dot{9};$$
$$137 = \Gamma(\sqrt{9!}) + 9 + 9 - \dot{9}; \qquad 139 = \Gamma(\sqrt{9!}) + 9 + 9 + \dot{9};$$
$$113 = \Gamma\Gamma 4 - \sqrt{4} + \cdot 4 - \sqrt{4}.$$

13237. (R. W. D. Christie.)—P, Q, R, act at a point parallel to the sides taken in order of an equilateral triangle, and the resultant is $b^2 + c^2 - bc$. Show that, while P is arbitrary, Q and R have each three separate values, and find them.

Solution by the Proposer.

The resultant $= (P_2 + Q^2 + R^2 - PQ - QR - RP)^{\frac{1}{2}}$,

and $\tan \mathrm{ROM} = [(Q-R)\sqrt{3}]/(2P-Q-R)$.

The three values are

$$P,\ P - b^2 + c^2,\ P + 2bc - b^2; \qquad P,\ P - 2bc + c^2,\ P - 2bc + b^2;$$
$$P,\ P + b^2 - c^2,\ P + 2bc - c^2.$$

Another value is $P = c^2$, $Q = b^2$, $R = 2bc$;

and, since we may add any constant a to P, Q, R, we also have

$$P = a + c^2, \quad Q = a + b^2, \quad R = a + 2bc, \quad \&c.$$

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6689. (J. R. HARRIS, M.A.)—A tetrahedron is cut by a plane so that the section shall be a rhombus. Prove that the side of the rhombus is half the harmonic mean between a pair of opposite edges.

Solution by W. J. GREENSTREET, M.A.

If ABCD be a tetrahedron, and EFGH a plane section parallel to AD and BC, E, F, G, H being on AB, AC, CD, DB respectively, then EF and GH are each parallel to BC, and EH is similarly parallel to FG. Also, since EFGH is a rhombus,

$$AC/AF = BC/EF; \quad \text{therefore} \quad (AC-AF)/AC = (BC-EF)/BC.$$

Also $$(AC-AF)/AC = FG/AD = EF/AD;$$

therefore $$(BC-EF)/EF = BC/AD;$$

therefore $$1/EF = 1/AD + 1/BC; \quad \text{therefore, \&c.}$$

The section must be parallel to two of the opposite edges, and only one section parallel to each pair give the relations for a rhombus; therefore only three planes can be drawn making a rhombus. If p_1, p_2, p_3 are the sides of these sections, the above relation gives us $\Sigma p^{-1} = \Sigma a^{-1}$, where a, b, c, d, e, f are the six edges.

13098. (R. W. D. CHRISTIE.)—Find a cube greater than unity of the form $\frac{1}{2}n(n+1)$.

Solution by the PROPOSER.

If $T_n = a^3$, then a is integral, for T_n always is whether n be odd or even. $\quad T_n = T_{2m}$ or $T_{2m-1} = 2m^2 \pm m$ (m integral).

(1) Let $$2m^2 + m = m(2m+1) = a^3;$$

therefore a cannot be prime. Let $a = bm$ where b may be integral or fractional; therefore

$$m(2m+1) = b^3m^3, \quad 2m+1 = b^3m^2, \quad 1 = b^3m^2 - 2m = m(b^3m-2);$$

which is impossible in integers.

(2) Similarly with $$2m^2 - m = a^3.$$

So that it would appear impossible to have $T_n = a^3$ greater than unity.

7340. (Professor ASUTOSH MUKHOPÂDHYÂY).—The maximum ellipsoid which can be inscribed in a given tetrahedron has its axes coincident with the mechanical principal axes through the centroid.

Mr. GREENSTREET, in regard to this Question, refers to the *Mathematician*, Vol. II. (1856), pp. 97–100.

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14573. (Professor E. J. NANSON.)—Prove that

$$\Sigma \sin^3 A \sin (B-C) = -\sin (A+B+C) \Pi \sin (B-C).$$

Solution by Rev. T. ROACH, M.A.; H. W. CURJEL, M.A.; *and* L. E. READ, B.A.

$$\Sigma \sin^3 A \sin (B-C)$$
$$= \Sigma \sin A \sin (B-C) - \Sigma \sin A \cos^2 A \sin (B-C) = -\Sigma \sin A \cos^2 A \sin (B-C)$$
$$= \sin A \sin B \cos C (\cos^2 B - \cos^2 A)$$
$$+ \cos A \cos B \sin C (\sin A \cos A - \sin B \cos B) - \sin C \cos^2 C \sin (A-B)$$
$$= \sin (A-B) \{\sin A \sin B \cos C \sin (A+B)$$
$$+ \cos A \cos B \sin C \cos (A+B) - \sin C \cos^2 C\}$$
$$= \sin (A-B) [\sin A \sin B \{\cos C \sin (A+B) + \sin C \cos (A+B)\}$$
$$+ (\cos A \cos B - \sin A \sin B) \sin C \cos (A+B) - \sin C \cos^2 C]$$
$$= \sin (A-B) [\sin A \sin B \sin (A+B+C) + \sin C \{\cos^2 (A+B) - \cos^2 C\}]$$
$$= \sin (A-B) \sin (A+B+C) \{\sin A \sin B + \sin C \sin (C-A-B)\}$$
$$= \sin (A-B) \sin (A+B+C) \{\sin A \sin (B-C+C) - \sin C \sin (B-C+A)\}$$
$$= \sin (A-B) \sin (A+B+C) \sin (B-C) \sin (A-C).$$

14477. (C. E. BICKMORE, M.A.)—Express as the product of two factors, each factor being the sum of two squares, $(a^2-4b^4+c^4)^4+(4abc)^4$.

Remarks by Professor K. J. SANJÁNA, M.A.

Two solutions of this Question are given in *Reprint*, Vol. LXXIII., p. 104. The first solution, though not of the form desired, may be practically useful. For instance, taking $a=1$, $b=\sqrt{2}$, $c=3$, we have $\alpha=66$, $\beta=12$; thus $66^4+(12\sqrt{2})^4=(54^2+12^2)(78^2+12^2)$, or, cancelling 36^2, $121^2+8^2=(9^2+2^2)(13^2+2^2)$.

In the second solution there is some mistake. The correct form is

$$(a^2-4b^4+c^4)^4+(4abc)^4 = \{(a^2-4ab^2+4b^4-c^4)^2+4a^2c^4\}$$
$$\times \{(a^2+4ab^2+4b^4-c^4)^2+4a^2c^4\}.$$

The following results are more symmetrical:—

$$(a^4-4b^4+c^4)^4+(4a^2bc)^4 = \{(a^4-4a^2b^2+4b^4-c^4)^2+4a^4c^4\}$$
$$\times \{(a^4+4a^2b^2+4b^4-c^4)^2+4a^4c^4\},$$

and a similar expression for $(a^4-4b^4+c^4)^4+(4abc^2)^4$. Putting $a^2=a'$, $b^2=b'$, $c^2=c'$, and suppressing accents, we shall have

$$(a^2-4b^2+c^2)^4+(4a\sqrt{bc})^4 = \{(a^2-4ab+4b^2-c^2)^2+4a^2c^2\}$$
$$\times \{(a^2+4ab+4b^2-c^2)^2+4a^2c^2\},$$

a formula easy of application. For instance, take $a=1$, $b=2$, $c=8$; we get $49^4+16^4=(55^2+16^2)(39^2+16^2)$.

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[The PROPOSER adds the following:—"As Professor SANJÁNA points out, there is a mistake in the second solution, viz., the omission of the term $-4b^2c^2\rho^2$ in lines 6 and 7 of page 105. The product $(x-\rho y)(x+\rho y)$ reduces to $(a^2-4ab^2+4b^4-c^4-2ac^2\rho^2)(a^2+4ab^2+4b^4-c^4+2ac^2\rho^2)$;

and the product $(x-\rho^2 y)(x+\rho^2 y)$ to

$$(a^2-4ab^2+4b^4-c^4+2ac^2\rho^2)(a^2+4ab^2+4b^4-c^4-2ac^2\rho^2);$$

whence

$$(x^4+y^4)=\left\{(a^2-4ab^2+4b^4-c^4)^2+(2ac^2)^2\right\}\times\left\{(a^2+4ab^2+4b^4-c^4)^2+(2ac^2)^2\right\}.$$

There is a further error in line 1, viz., '$4b^2c^2\rho$' for '$4b^2c^2\rho^2$' in the second term of the second expression in brackets."]

14296. (H. MACCOLL, B.A.)—If x, y, z be each taken at random between 0 and 1, show that the chance that the fraction $\frac{z(1-x-y)}{1-y-yz}$ will also be between 0 and 1 is $\frac{4}{3}-\log_e 2$, which is very nearly 5/9.

Solution by the PROPOSER.

Let the statement Q assert that the event in question will happen; and let A assert the data, or given conditions. The chance required is

$$(AQ\int dx\int dy\int dz)/(A\int dx\int dy\int dz,$$

in which A and Q define the limits of integration.

Now

$A = x_{1'.0}y_{1'.0}z_{1'.0}$,

so that the corresponding definite integral is unity. Hence the required chance is

$x_1=1$	$y_1=1$	$z_1=1$
$x_2=1-y$	$y_2=1/(1+z)$	
$x_3=(y+z-1)/z$	$y_3=1-z$	

$$AQ\int dx\int dy\int dz.$$

Let N denote the numerator and D the denominator of the fraction $z(1-x-y)/(1-y-yz)$; and let the symbols u, v (as exponents) assert *positiveness* and *negativeness* respectively. We get

$$Q = N^uD^u(N-D)^v+N^vD^v(N-D)^u,$$

$$N^u=(1-x-y)^u=\left\{x-(1-y)\right\}^v=x_{2'},$$

$$D^u=(1-y-yz)^u=\left\{y(1+z)-1\right\}^v=\left\{y-1/(1+z)\right\}^v=y_{2'},$$

$$(N-D)^v=(z-xz-1+y)^v=\left\{x-(y+z-1)/z\right\}^u=x_3,$$

$$(N-D)^u=(x_{3'},\quad N^v=x_2,\quad D^v=y_2.$$

Hence $$Q=x_{2'.3}y_{2'}+x_{3'.2}y_2.$$

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To find AQ, we multiply successively by $x_{1'.0}$, $y_{1'.0}$, $z_{1'.0}$.

$$Qx_{1'.0} = x_{2'.1'.3.0}y_{2'} + x_{3'.1'.2.0}y_2.$$

But $x_{2'.1'} = x_{2'}$, $x_{3.0} = x_3y_3 + x_0y_{3'}$, $x_{3'.1'} = x_{3'}$, and $x_{2.0} = x_2$; so that

$$Qx_{1'.0} = x_{2'.3}y_{2'.3} + x_{2'.0}y_{2'.3'} + x_{3'.2}y_2.$$

The formula $x_{m'.n} = x_{m'.n}(x_m - x_n)^u$, introduces here no fresh statements in y; for

$$x_{2'.3} = x_{2'.3}(x_2 - x_3)^u = x_{2'.3}y_{2'};$$

$$x_{2'.0} = x_{2'.0}(x_2 - 0)^u = x_{2'.0}\epsilon \quad (\epsilon \text{ denotes a certainty});$$

$$x_{3'.2} = x_{3'.2}(x_3 - x_2)^u = x_{3'.2}y_2.$$

Multiplying next by $y_{1'.0}$, we get

$$Qx_{1'.0}y_{1'.0} = x_{2'.3}y_{2'.1'.3.0} + x_{2'.0}y_{2'.3'.1'.0} + x_{3'.2}y_{1'.2.0}.$$

But $y_{2'.1'} = y_{2'}$, $y_{3.0} = y_3$, $y_{2'.3'.1'} = y_{3'}$, and $y_{2.0} = y_2$, so that

$$Qx_{1'.0}y_{1'.0} = x_{2'.3}y_{2'.3} + x_{2'.0}y_{3'.0} + x_{3'.2}y_{1'.2}.$$

From $y_{2'.3}$, $y_{3'.0}$, $y_{1'.2}$ we get no fresh statements in z. Hence, finally,

$$Qx_{1'.0}y_{1'.0}z_{1'.0} = (x_{2'.3}y_{2'.3} + x_{2'.0}y_{3'.0} + x_{3'.2}y_{1'.2})z_{1'.0}.$$

This is the final value of AQ, and determines the limits of integration in the order of variation x, y, z. The integrations are easy, and the result is $\frac{1}{4} - \log_e 2$.

[For a full explanation of this method of finding the limits of integration in multiple integrals, see the PROPOSER's first paper on the "Calculus of Equivalent Statements" in the *Proc. of the Lond. Math. Soc.*]

14379. (J. J. BARNIVILLE, B.A.)—Having given $u_{n-1} + u_{n+1} = 4u_n$, prove that

$$\frac{1}{1^2+3} + \frac{1}{5^2+3} + \frac{1}{19^2+3} + \frac{1}{71^2+3} + \ldots = \frac{1}{2\sqrt{3}};$$

$$\frac{1}{1^2+8} + \frac{1}{5^2+8} + \frac{1}{19^2+8} + \frac{1}{71^2+8} + \ldots = \frac{1}{4\sqrt{3}};$$

$$\frac{1}{5^2-1} + \frac{1}{19^2-1} + \frac{1}{71^2-1} + \frac{1}{265^2-1} + \ldots = \frac{1}{3} - \frac{1}{2\sqrt{3}};$$

$$\frac{1}{5^2-6} + \frac{1}{19^2-6} + \frac{1}{71^2-6} + \frac{1}{265^2-6} + \ldots = \frac{1}{5} - \frac{1}{4\sqrt{3}}.$$

Solution by G. D. WILSON, B.A.; *and* Professor SANJÁNA.

Let x_n, y_n, z_n, u_n stand for the nth terms of the series

1, 2, 7, ...; 1, 3, 11, ...; 1, 4, 15, ...; 1, 5, 19, ...

..... $(\alpha, \beta, \gamma, \delta)$.

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Then $x_n = \frac{1}{3}(u_n - u_{n-1})$; $y_n = \frac{1}{6}(u_{n+1} - u_{n-1})$; $z_n = \frac{1}{6}(u_{n+1} - u_n)$.

Now in series for which $4w_n = w_{n+1} + w_{n-1}$ ..... (1),

we have $14w_n = w_{n+2} + w_{n-2}$, $w_{n+1}w_{n-1} - w_n^2 = \text{constant}$ ..... (2, 3),

$w_{n+2}w_{n-2} - w_n^2 = 16(w_{n+1}w_{n-1} - w_n^2)$ ..... (4),

$w_{n+1}w_{n-2} - w_n w_{n-1} = \text{constant}$ ..... (5).

From (1) and (3), $4x_n x_{n+1} = 2u_n^2 + 6$;

from (2) and (4), $36y_{n+1}y_{n-1} = 12u_n^2 + 96$;

from (1) and (3), $36z_n z_{n-1} = 6u_n^2 - 6$;

and from (3), $u_{n-1}u_{n+1} = u_n^2 - 6$;

and hence, from (3),

$$1/(u_n^2 + 3) = \tfrac{1}{6}\{x_{n-1}/x_n - x_n/x_{n+1}\};$$

from (5), $1/(u_n^2 + 8) = \frac{1}{24}\{y_{n-2}/y_{n-1} - y_n/y_{n+1}\}$;

from (3), $1/(u_n^2 - 1) = -\frac{1}{8}\{z_{n-2}/z_{n-1} - z_{n-1}/z_n\}$;

and from (5), $1/(u_n^2 - 6) = -\frac{1}{24}\{u_{n-2}/u_{n-1} - u_n/u_{n+1}\}$.

For all these series, the limit, when n is infinite, of w_n/w_{n+1} is $2 - \sqrt{3}$. The values of the given series can now be immediately obtained.

12616. (Professor Schmitt.)—Prove that the product of two numbers, each the sum of four squares, may be expressed as the sum of four squares in 48 different ways, and write some or all of the 48 ways.

Solution by Professors Sanjána, Colaw, Bhattacharyya, *and others.*

It may easily be proved that

$$\left|\left|\begin{matrix} a, & b, & c, & d \\ a', & b', & c', & d' \end{matrix}\right|\right|^2 = \{(bc') + (ad')\}^2 + \{(ca') + (bd')\}^2 + \{(ab') + (cd')^2$$

(see Salmon's *Higher Algebra*, § 24, *et seq.*). The left side is

$$(a^2 + b^2 + c^2 + d^2)(a'^2 + b'^2 + c'^2 + d'^2) - (aa' + bb' + cc' + dd')^2;$$

thus we get $(a^2 + b^2 + c^2 + d^2)(a'^2 + b'^2 + c'^2 + d'^2)$

$= \{(bc') + (ad')\}^2 + \{(ca') + (bd')\}^2 + \{(ab') + (cd')\}^2 + (aa' + bb' + cc' + dd')^2$.

If we write $\left|\left|\begin{matrix} a', & b', & c', & d' \\ a, & b, & c, & d \end{matrix}\right|\right|$ for the original expression, the same product is found:

$= \{(b'c) + (a'd)\}^2 + \{(c'a) + (b'd)\}^2 + \{(a'b) + (c'd)\}^2 + (a'a + b'b + c'c + d'd)^2$.

Thus the number of ways is evidently double of the number of permutations of the three letters a, b, c, when d or d' occurs in the second terms of the first three brackets on the right side; similarly when this letter is c or c', b or b', a or a'.

The total number of ways is therefore $4 \times 2 \times 6 = 48$.

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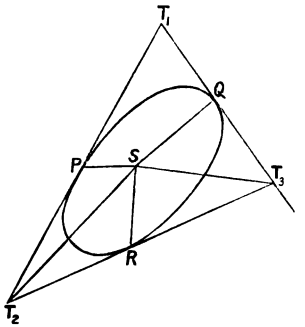
14622. (Rev. T. ROACH, M.A.)—If $T_1T_2T_3$ is a triangle described about an ellipse, then the angle subtended by T_2, T_3 at $S = \frac{1}{2}$ complement of that subtended by P, Q, the points where the tangents from T_1 touch. [" Complement " here denotes $2\pi - PSQ$.]

Solution by the PROPOSER.



$\angle T_2SR = T_2SP$,

$\angle T_3SR = T_3SQ$;

therefore

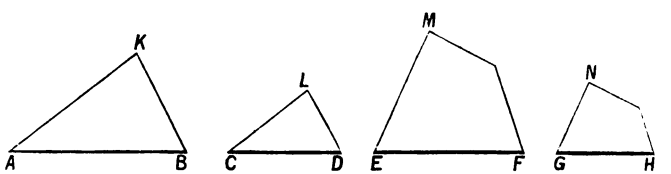
$T_2ST_3 = \frac{1}{2}(PSR + RSQ)$

$= \frac{1}{2}$ complement of PSQ.

A new Proof of Euclid, Book VI., Prop. 22, *Theorem. By* M. R. PEIRIS.

If four straight lines be proportional, and a pair of similar rectilineal figures be similarly described on the first and second, and also a pair on the third and fourth, these figures shall be proportional.

Conversely, if a rectilineal figure on the first of four straight lines be to the similar and similarly described figure on the second as a rectilineal figure on the third is to the similar and similarly described figure on the fourth, the four straight lines shall be proportional.



Let AB, CD, EF, GH be proportionals, so that AB : CD = EF : GH. Let similar figures KAB, LCD be similarly described on AB, CD, and also let similar figures MF and NH be similarly described on EF, GH. Then shall the fig. KAB : the fig. LCD = the fig. MF : the fig. NH.

Since AB : CD = EF : GH,

the duplicate ratio of AB : CD = the duplicate ratio of EF : GH (v. 16). But, the duplicate ratio of AB : CD is the ratio of the fig. KAB to the fig. LCD (VI. 19). Also, the duplicate ratio of EF : GH is the ratio o the fig. MF to the fig. NH (VI. 20). Therefore the fig. KAB : the fig. LCD = the fig. MF : the fig. NH.

Conversely, let the fig. KAB : the fig. LCD = the fig. MF : the fig. NH. Then shall AB : CD = EF : GH. For, the fig. KAB : the

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fig. LCD in the duplicate ratio of AB : CD (VI. 19). Also, the fig. MF : the fig. NH in the duplicate ratio of EF : GH (VI. 20). Therefore the duplicate ratio of AB : CD = the duplicate ratio of EF : GH. Therefore AB : CD = EF : GH (V. 16).

[Mr. PEIRIS says that, as far as he can see, the Editors of Euclid have failed to make use of Book V. in this proposition, thus rendering the proof long and cumbrous. SIMSON and those who have copied him, as noted by CASEY, so word the theorem as to require all four figures to be similar, which need not be.]

14750. (R. P. PARANJPYE, B.A.)—Find the limit, if there be any, of

$$4^{-4^{-4^{\text{ad inf.}}}}$$

Solution by G. M.

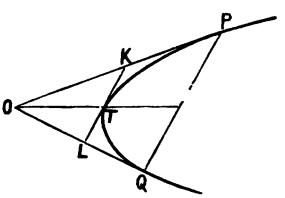
Let $x = 4^{-4^{-4^{\text{ad inf.}}}}$; then $x = 4^{-x} = 1/4^x$; therefore $x \cdot 4^x = 1$.

From this it is clear that x must be positive, also that it must lie between 0 and 1; and, by inspection, $x = \frac{1}{2}$ will satisfy the equation; therefore limit of $4^{-4^{-4^{\cdots}}} = \frac{1}{2}$.

14617. (V. DANIEL.)—The locus of the middle point of the intercept of a variable tangent by tangents from a fixed point to a parabola is the tangent parallel to the polar of the fixed point.

Solution by LIONEL E. REAY, B.A.; *and* W. L. THOMSON.

The tangent KL, which is parallel to PQ, meets the line at infinity on PQ at the point Ω, say. Then O and Ω are conjugate points. Hence any tangent to the parabola is divided harmonically by OP, OQ, KL and line at infinity. Therefore its intercept between OP and OQ is bisected by KL.



14623. (Professor G. B. MATHEWS, F.R.S.)—Prove that

$$\Sigma \mathrm{E}\left(\tfrac{2}{5}n\right) x^n = (x^3 + x^5)/\{(1-x)(1-x^5)\},$$

and show how to find a similar expression for $\Sigma \mathrm{E}\{an/(a+b)\}\, x^n$. [As usual, $\mathrm{E}(x)$ is the integer defined by $\mathrm{E}(x) \leqslant x < \mathrm{E}(x) + 1$.]

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Solution by G. D. WILSON.

$$\sum_{n=1}^{\infty} \mathrm{E}\left(\frac{an}{a+b}\right) x^n = \sum_{\lambda=1}^{\infty} \left\{ \sum_{n=1+\mathrm{E}[\lambda(a+b)/a]}^{\mathrm{E}[(\lambda+1)(a+b)/a]} \lambda x^n + x^{\lambda(a+b)} \right\}$$

$$= \frac{x^{a+b}}{1-x^{a+b}} + \sum_{\lambda=1}^{\infty} \lambda \frac{x^{\lambda-1+\mathrm{E}(\lambda b/a)} - x^{\lambda+2+\mathrm{E}[(\lambda+1)b/a]}}{1-x}$$

$$= \frac{x^{a+b}}{1-x^{a+b}} + \frac{x}{1-x} \sum_{\lambda=1}^{\infty} x^{\lambda+\mathrm{E}(\lambda b/a)}$$

$$= \frac{x^{a+b}}{1-x^{a+b}} + \frac{x}{1-x} \sum_{\lambda=1}^{a} x^{\lambda+\mathrm{E}(\lambda b/a)} \left\{1 + x^{a+b} + x^{2(a+b)} \ldots\right\}$$

$$= \frac{1}{1-x^{a+b}} \left\{ x^{a+b} + \frac{x}{1-x} \sum_{\lambda=1}^{a} x^{\lambda+\mathrm{E}(\lambda b/a)} \right\}.$$

Hence $\Sigma \mathrm{E}\left(\frac{2}{5}n\right) x^n = \frac{1}{1-x^5}\left\{x^5 + \frac{x}{1-x}(x^2+x^5)\right\} = \frac{x^3+x^5}{(1-x)(1-x^5)}$.

14592. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Solve in integers in a general form the equation $N_1 . N_3 = N_2 . N_4$, where $N_r = x_r^4 + 4y_r^4$, and give numerical examples.

Solution by the PROPOSER.

Numbers of the form $N_r = x_r^4 + 4y_r^4$ have been studied under the name of *Bin-Aurifeuillians* in the PROPOSER's paper "On Aurifeuillians" in *Lond. Math. Soc. Proc.*, Vol. XXIX. Two of these, say N_3, N_4, may be formed (as in Art. 19, *op. cit.*) so as to contain the *common* Aurifeuillian factor (a^2+b^2), by writing $x_3 = a-b$, $x_4 = a+b$, $y_3 = y_4 = b$. Then

$$\frac{N_4}{N_3} = \frac{x_4^4 + 4y_4^4}{x_3^4 + 4y_3^4} = \frac{(a+b)^4 + 4b^4}{(a-b)^4 + 4b^4} = \frac{(a+2b)^2 + b^2}{(a-2b)^2 + b^2},$$

after cancelling the common factor (a^2+b^2); and this should $= N_1 + N_2$.
Therefore $(a+2b)^2 + b^2 = N_1 = x_1^4 + 4y_1^4$, $(a-2b)^2 + b^2 = N_2 = x_2^4 + 4y_2^4$;
therefore $b = 2y_1^2 = 2y_2^2$, $a+2b = x_1^2$, $a-2b = x_2^2$;
therefore $a = \frac{1}{2}(x_1^2 + x_2^2)$, $4b = x_1^2 - x_2^2 = 2(2y_1)^2 = 2(2y_2)^2$.
The *general* solution of this Diophantine in terms of two arbitraries (t odd) is $x_1 = t^2 + 2u^2$, $x_2 = t^2 \sim 2u^2$, $y_1 = y_2 = tu$.
Hence $a = t^4 + 4u^4$, $b = 2t^2u^2$;
therefore $x_3 = t^4 - 2t^2u^2 + 4u^4$, $x_4 = t^4 + 2t^2u^2 + 4u^4$, $y_3 = y_4 = 2t^2u^2$.

Finally, $N_1 = (t^2+2u^2)^4 + 4(tu)^4$, $N_3 = (t^4 - 2t^2u^2 + 4u^4)^4 + 4(2t^2u^2)^4$,
$N_2 = (t^2 \sim 2u^2)^4 + 4(tu)^4$, $N_4 = (t^4 + 2t^2u^2 + 4u^4)^4 + 4(2t^2u^2)^4$.

Ex.—Take $t = 1$, $u = 1$; then
$x_1 = 3$, $x_2 = 1$; $y_1 = y_2 = 1$; $x_3 = 3$, $x_4 = 7$; $y_3 = y_4 = 2$.
$N_1 = 3^4 + 4.1^4$, $N_2 = 1^4 + 4.1^4$, $N_3 = 3^4 + 4.2^4$, $N_4 = 7^4 + 4.2^4$;
$N_1 . N_3 = 85 . 145 = 5 . 2465 = N_3 . N_4$.

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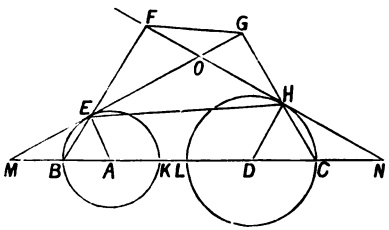
14583. (J. Macleod, M.A.)—EBK, HCL, two unequal circles external to each other, are cut by the line of centres AD in B, C. AM = diameter of EBK, and DN = diameter of HCL; M, N being on line of centres. ME is tangent to EBK, and NH to HCL. EB and NH meet in F, and ME and HC in G. Prove that FG = ½EH.

Solution by K. G. Panchapagesa Aiyar; Constance I. Marks *and others.*

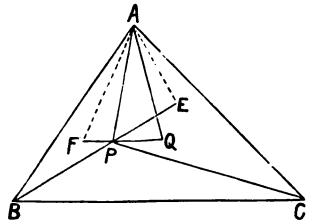


Let EG and HF intersect at O. Join AE and DH. In the right-angled triangle DHN, CD = CH = CN, and the triangle CDH is equilateral; therefore ∠CNH = ⅙π. Again, since the triangle ABE is equilateral, ∠ABE = ⅓π; therefore ∠BFN = ½π. Similarly, ∠CGM = ½π; therefore E, F, G, H are concyclic; therefore the triangles FOG and EOH are similar; therefore

FG : GO :: EH : HO. But GO = ½OH; therefore FG = ½EH.

14589. (Rev. J. Cullen.)—If α, β, γ are vectors from a point P to the vertices of a given triangle ABC; also p, q, r are the trilinear coordinates of P, and p_1, q_1, r_1 those of another point Q, show that the vector PQ is $\Sigma ap_1\alpha/\Sigma ap_1$, with the condition $\Sigma apa = 0$. [*E.g.*, $\Sigma a^2\alpha/\Sigma a^2$ with $\Sigma\alpha = 0$ is the vector from the centroid to the symmedian point.] Apply the above result to prove Quest. 14532.

Note by Professor K. J. Sanjána, M.A.



The first part is readily solved by statical principles. Let forces along PA, PB, PC be proportional to PA.△PBC, PB.△PCA, PC.△PAB, respectively; then sum of the moments about A

= PB.△PCA.AE + PC.△PAB.AF

= 2△PCA.△PBA + 2△PAB.△PCA

= 0.

So also the moments vanish round B and C; hence the forces are in equilibrium, *i.e.*, $\Sigma apa = 0$. So, if the vectors of Q be α_1, β_1, γ_1, we have $\Sigma a_1p_1a = 0$. Now PQ = PA + AQ; therefore

$$PQ.ap_1 = PA.ap_1 + AQ.ap_1;$$

hence $$PQ\,(ap_1 + bq_1 + cr_1) = \Sigma ap_1\alpha - \Sigma a_1p_1a = \Sigma ap_1\alpha;$$

so that $$PQ = \Sigma ap_1\alpha \div \Sigma ap_1.$$

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9669. (Professor ABINASH CHANDRA BASU, M.A.)—Show that the equations $2x^4+3bx^3+6cx^2+dx-6e=0$, $x^4+bx^3+cx^2+e=0$ are so related that, if they have a common root, that root will be a double root of the first. If the roots of the first are m, n, p, and q, express the latter equation in the form of a determinant equation of the fourth order.

Solution by H. J. WOODALL, A.R.C.S.

If (1) and (2) have a common root, this must be a root of $\{(1)+(2)\times 6\}$, *i.e.*, of $8x^4+9bx^3+12cx^2+dx=0$.....(3);

but this is x [first derived equation of (1)] $=0$. Therefore this common root will be, also, a double root of (1).

To find the determinant form of (2) in terms of x and the four roots of (1). Substitute m, n, p, q for x in (1), and eliminate $1:b:c:d:-e$ between these four new equations and (2); we get

$$\begin{vmatrix} x^4, & x^3, & x^2, & 0, & 1 \\ 2m^4, & 3m^3, & 6m^2, & m, & 6 \\ 2n^4, & 3n^3, & 6n^2, & n, & 6 \\ 2p^4, & 3p^3, & 6p^2, & p, & 6 \\ 2q^4, & 3q^3, & 6q^2, & q, & 6 \end{vmatrix} = 0,$$

which can be easily reduced to

$$\begin{vmatrix} 3x^4, & 2x^3, & x^2, & m \\ n(n^3-m^3), & n(n^2-m^2), & n(n-m), & n-m \\ p(p^3-m^3), & p(p^2-m^2), & p(p-m), & p-m \\ q(q^3-m^3), & q(q^2-m^2), & q(q-m), & q-m \end{vmatrix} = 0;$$

$$\begin{vmatrix} 3x^4, & 2x^3, & x^2, & m \\ n(n^3-m^3), & n(n^2-m^2), & n(n-m), & n-m \\ p(p^3-m^3), & p(p^2-m^2), & p(p-m), & p-m \\ q(q^3-m^3), & q(q^2-m^2), & q(q-m), & q-m \end{vmatrix} = 0;$$

therefore

$$\begin{vmatrix} 3x^4 & 2x^3 & x^2, & m, \\ n(n^2+nm+m^2), & n(n+m), & n, & 1, \\ p(p^2+pm+m^2), & p(p+m), & p, & 1, \\ q(q^2+qm+m^2), & q(q+m), & q, & 1, \end{vmatrix} = 0;$$

therefore

$$\begin{vmatrix} 3x^4-2m.x^3, & 2x^3-m.x^2, & x^2, & m \\ n^3, & n^2, & n, & 1 \\ p^3, & p^2, & p, & 1 \\ q^3, & q^2, & q, & 1 \end{vmatrix} = 0.$$

12382. (W. J. C. MILLER, B.A.)—Construct a triangle of given species which shall have its vertices (1) on two concentric circles; and (2) generalize the problem.

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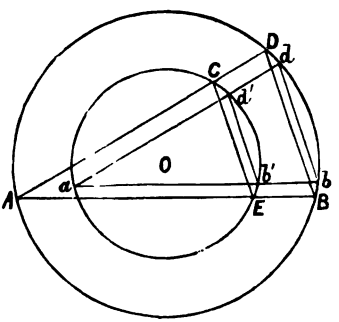
Solution by the late MORGAN BRIERLEY.



(1) In the outer of the two concentric circles (O), construct the triangle ABD, of the given species, by a known method, the sides AB, AD intersecting the inner circle in the points E, C, respectively, adjacent to B, D. Join E, C, and, obviously, the triangle AEC is similar to ABD and of the given species, and its vertices upon two concentric circles.

(2) When two of the vertices b, d are on the outer circle, the triangle $ab'd'$, given in species, requires to be inscribed in the inner circle, and the sides produced to the outer one.

6726. (J. HAMMOND, M.A.)—If

$$S = \Gamma(r+1) + \frac{\Gamma(r+2)}{2!} + \frac{\Gamma(r+3)}{2!\,3!} + \frac{\Gamma(r+4)}{3!\,4!} + \dots \text{ to } \infty,$$

prove that $$S = \frac{\Gamma(r)\sin r\pi}{\pi} \int_0^1 e^x \left(\frac{x}{1-x}\right)^r dx.$$

Solution by Rev. J. CULLEN.

Since $$\Gamma(m)\,\Gamma(n)/\Gamma(m+n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx,$$

we have

$$\Gamma(m)/(m-r-1)!\,\Gamma(m+1-r) = \{\Gamma(1-r)\}^{-1} \int_0^1 \frac{x^{m-r-1}}{(m-r-1)!} \left(\frac{x}{1-x}\right)^r dx.$$

Putting $m = r+1$, $r+2$, &c., and summing, we get

$$S = \{\Gamma(1-r)\}^{-1} \int_0^1 e^x \left(\frac{x}{1-x}\right) dx;$$

but $$\Gamma(1-r)\,\Gamma(r) = \pi/\sin r\pi.$$

Hence the result.

14610. (R. F. DAVIS, M.A.)—Prove that the length of the chord of the circumcircle to a triangle ABC, drawn so as to bisect the sides AB, AC, is $2R\{1-\cos^2 B\cos^2 C\}^{\frac{1}{2}}$.

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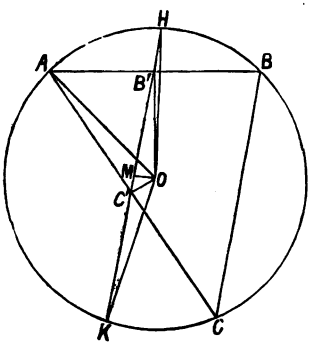
Solution by JOHN LEWIS; CONSTANCE I. MARKS, B.A.; *and others.*



B′ and C′ are the middle points of AB and AC, respectively; O is the centre of the circumcircle. Therefore OB′, OC′ are perpendicular to AB, AC, respectively.

Let B′C′ cut the circle in H and K. Join OH, OK. Draw OM perpendicular to B′C′. Then

$$HM = KM.$$

The points A, B′, O, C′ are cyclic; therefore $\angle B'AO = \angle B'C'O$.

$$B'O = R \sin B'AO$$
$$= R \sin B'C'O = R \cos C;$$

also $OM = B'O \sin MB'O$

$$= R \cos B \cos C;$$

therefore $HK = 2(R^2 + OM^2) = 2R(1 - \cos^2 B \cos^2 C)^{\frac{1}{2}}$.

[Professor JAN DE VRIES, Ph.D., solves the Question as follows:—

The distance d of that chord to the circumcentre is equal to $\frac{1}{2}h_a - R \cos A$, h_a being the length of the perpendicular from A to BC, whereas $R \cos A$ is the distance of BC to the circumcentre. Since

$$h_a = b \sin C = 2R \sin B \sin C,$$
$$d = R \sin B \sin C + R \cos (B + C) = R \cos B \cos C.$$

Therefore the length of the chord will be $2R(1 - \cos^2 B \cos^2 C)^{\frac{1}{2}}$.]

14530. (ROBERT W. D. CHRISTIE.)—If $6m + 1$ = a prime, prove

$$r^{2m} + r_1^m \equiv 0 \pmod{p} \quad \text{and} \quad r_1^{2m} + r^m \equiv 0 \pmod{p},$$

where r and r_1 are associated primitive roots.

Solution by the PROPOSER; Lt.-Col. ALLAN CUNNINGHAM, R.E.; *and* H. J. WOODALL, A.R.C.S.

We have $r^{3m} + 1 = 0 \pmod{6m+1}$ and $rr_1 = 1 \pmod{6m+1}$, by the general theory; therefore

$$r^m r_1^m = 1 \pmod{6m+1} \quad \text{and} \quad r^{3m} + r^m r_1^m = 0 \pmod{6m+1},$$

i.e., $r^m (r^{2m} + r_1^m) = 0 \pmod{6m+1}$; thus $r^{2m} + r_1^m = 0 \pmod{6m+1}$, where r_1, r are interchangeable.

This modular equation should be compared with others of the PROPOSER, viz.,

$$r^m + r_1^m = 1 \pmod{6m+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$
$$r^m + r_1^m = 0 \pmod{4m+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3);$$

and the more general ones (*v.* Quest. 14668), n being *any* integer,

$$r^{2m-n} + r_1^n = 0 \pmod{4m+1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4),$$
$$r^{2m-n-1} + r_1^n = 0 \pmod{4m-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5).$$

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14552. (D. BIDDLE.)—It being given that $N = 2\Delta M + 1$, and that its factors are of the form $2\Delta m + 1$, prove the validity of the method of factorization employed in the adjoining example, the rule being as follows:—Take, as first divisor (D_1), the number next below $\sqrt{N}$ of proper form, say $2\Delta m_1 + 1$; and take, as first dividend, $M - m_1$. These yield, as quotient and remainder, Q_1 and R_1. Since divisors patently composite can be discarded, take, in succession, $D_n = D_{n-1} - 2\Delta\mu_n$, where $\mu_n = 1$ or other small digit that does not carry D_n beyond the next prime (of the particular form) in descending order. Next, utilize

$N = 150809, \quad \Delta = 7.$

10745	(μ)	(κ)	(Q)	(R)
379			28	133
337	3	3	31	301
323	1	2	33	90
281	3	5	38	74
253	2	4	42	128
239	1	3	45	0

$2 \cdot 7 \cdot 45 + 1 = 631;$

$\therefore \quad N = 239 \cdot 631.$

$$(2\Delta\mu_n Q_{n-1} + R_{n-1} + \mu_n) / D_n = \kappa_n$$

with R_n; also $\kappa_n + Q_{n-1} = Q_n$. When $R_x = 0$, $N = D_x(2\Delta Q_x + 1)$. [In the example, two divisors are tried which might have been excluded, namely, $323 = 17 \cdot 19$ and $253 = 11 \cdot 23$; but the final result is unaffected thereby.]

Solution by Rev. J. CULLEN.

We have $M - m_1 = Q_1 D_1 + R_1 = (N-1)/2\Delta - m_1;$

therefore $N = (2\Delta m_1 + 1)(2\Delta Q_1 + 1) + 2\Delta R_1,$

or $N = D_1(2\Delta Q_1 + 1) + 2\Delta R_1;$

also $N = D_2(2\Delta Q_2 + 1) + 2\Delta R_2.$

Now assume $N = D_{x-1}(2\Delta Q_{x-1} + 1) + 2\Delta R_{x-1},$

observing that $2\Delta\mu_x Q_{x-1} + R_{x-1} + \mu_x = D_x\kappa_x + R_x,$

i.e., $2\Delta\mu_x(2\Delta Q_{x-1} + 1) + 2\Delta R_x = 2\Delta(Q_x - Q_{x-1}) + 2\Delta R_x$

(by multiplying across by 2Δ). Now, using $D_x = D_{x-1} - 2\Delta\mu_x$, we have

$$D_x(2\Delta Q_x + 1) + 2\Delta R_x = D_{x-1}(2\Delta Q_{x-1} + 1) + 2\Delta R_{x-1};$$

therefore $N = D_x(2\Delta Q_{x+1})$, if $R_x = 0$.

14577. (Professor LANGHORNE ORCHARD, M.A., B.Sc.)—Sum to n terms the series $1 + 5 + 3 + 4 + 3\frac{1}{2} + 3\frac{3}{4} + 3\frac{5}{8} + \ldots$.

I. *Solution by* Professor SANJÁNA, M.A.; Rev. T. ROACH, M.A.; *and others.*

The series of first differences, $4, -2, 1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$ is a G.P. Thus the nth term is of the form $a + b(-\frac{1}{2})^{n-1}$; a and b are easily found to be $\frac{11}{3}$ and $-\frac{8}{3}$ respectively. Hence the sum

$$= \tfrac{11}{3}n - \tfrac{8}{3}(1 - \tfrac{1}{2} + \tfrac{1}{4} - \ldots) = \tfrac{11}{3}n - \tfrac{8}{3}\left\{1 - (-\tfrac{1}{2})^n\right\} / \left\{1 - (-\tfrac{1}{2})\right\}$$

$$= \tfrac{11}{3}n - \tfrac{16}{9}\left\{1 - (-\tfrac{1}{2})^n\right\} = \tfrac{1}{9}\left\{33n - 16 + \frac{(-1)^n}{2^{n-4}}\right\}.$$

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II. *Solution by* J. J. BARNIVILLE, B.A.

$$u_n + u_{n+1} = 2u_{n+2};$$

therefore $(2-x-x^2)(1+5x+3x^2+4x^3+\ldots) = 2+9x,$

i.e., $$1+5x+3x^2+4x^3+\ldots = \frac{2+9x}{(2+x)(1-x} = \frac{11}{3(1-x)} - \frac{16}{3(2+x)}.$$

Hence, equating coefficients

$$3u_n = 11+(-2)^{4-n},$$

and $$3s_n^1 = 11n-8\,(1-\tfrac{1}{2}+\tfrac{1}{4}-\tfrac{1}{8}+\ldots \text{ to } n \text{ terms}).$$

It may be noted that $u_\infty = 3\frac{2}{3}$.

14595. (R. KNOWLES.)—If d denote the distance between the centres of the nine-point and circum-circles of a triangle, prove that

$$4d^2 = R^2\,(1-8\cos A\cos B\cos C).$$

Solution by W. J. GREENSTREET, M.A., *and many others.*

With the usual lettering, N being the mid-point of OH,

$$4NO^2 = OH^2 = R^2\,(1-8\cos A\cos B\cos C).$$

14521. (D. BIDDLE.)—[In continuation of Quests. 14457 and 14490.] Writing down N in terms of 3 and its powers, the only coefficients used being ± 1 or 0, it will be convenient to omit both index and root, using only the sign preceding (or 0), and reckoning the powers from the right hand (as usual). Thus, $1843 = 3^7-3^5-3^4+3^2-3+1$, and it may be written $+0--0+-+$. Similarly, 1769 may be written $+-++0---$. Since the last sign is for $3^0\,(=1)$, the first index is 1 less than the number of signs (including zero). Prove that, having bracketed together any number of the given signs beginning with the first (which is always positive), and having assigned to them the value they would have if standing alone, we can, to the successive signs beyond, append p, p^2, p^3, ... in ascending order, and thus obtain a multiple of $3p\pm1$, a factor of N (p being even), provided we reverse the signs attached to the odd powers of p when the factor is $3p+1$, but leave them unchanged when the factor is $3p-1$. Thus, $1769 = (+-++0)---$, and $66+20-400+8000 = M\,(61)$; also $66-10-100-1000 = M\,(29)$.

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

The two properties in question may be proved in a *general* manner for any scale of notation (with slight modifications).

Let $$N = (a_n r^n+\ldots+a_{x+1}r^{x+1})+(a_x r^x+\ldots+a_1 r+a_0)$$

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be the expression of N in the scale whose radix is r, where the coefficients may be either $\pm$ (but all $< r$), except that the leading one a_n is $+$. Then, dividing N into two parts, as shown by the brackets, the number may be written $\quad N = X.r_{x+1} + N_x$ (N_x being the part to the right), and the new number (N′), formed as directed, is

$$\begin{aligned} N' &= X + (a_x p + a_{x-1} p^2 + \ldots + a_{x-y} p^{y+1} + \ldots + a_1 p^x + a_0 p^{x+1}) \\ &= (N - N_x)/r^{x+1} + \Sigma_y (a_{x-y} p^{y+1}) = 1/r^{x+1}.\{N + r^{x+1} \Sigma_y (a_{x-y} p^{y+1}) - N_x\} \\ &= 1/r^{x+1}.\{N + \Sigma_y (r^{x+1} a_{x-y} p^{y+1} - a_{x-y} r^{x-y})\} \\ &= 1/r^{x+1}.\left[N + \Sigma_y \{r^{x-y} a_{x-y} (r^{y+1} p^{y+1} - 1)\}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i.)}. \end{aligned}$$

Now $(r^{y+1} p^{y+1} - 1)$ is divisible by $(rp - 1)$ for *all* values of y; so that N′ is divisible by $(rp - 1)$ when the latter is a factor of N.

Next, let another number N″ be formed by writing $(-p)$ instead of p in N′ (this amounts to changing the signs of the coefficients a_{x-y} when y is even). Then

$$N'' = 1/r^{x+1}.\left[N + \Sigma_y \{r^{x-y} a_{x-y} [r^{y+1} (-p)^{y+1} - 1]\}\right] \ldots\ldots \text{(ii.)}.$$

Here $[r^{y+1}(-p)^{y+1} - 1]$ is divisible by $(rp + 1)$ for *all* values of y; so that N′ is divisible by $(rp + 1)$ when the latter is a factor of N.

Taking $r = 3$, the coefficients a_x all reduce to ± 1; and (i.), (ii.) reduce to the theorems required.

14320. (J. J. Barniville, B.A.)—Prove that

$$\frac{1}{1+2} + \frac{1}{13+2} + \frac{1}{233+2} + \ldots = \frac{\sqrt{5}+1}{8};$$

$$\frac{1}{2+2} + \frac{1}{34+2} + \frac{1}{610+2} + \ldots = \frac{\sqrt{5}}{8};$$

$$\frac{1}{5+2} + \frac{1}{89+2} + \frac{1}{1597+2} + \ldots = \frac{\sqrt{5}-1}{8};$$

$$\frac{1}{1+5} + \frac{1}{2+5} + \frac{1}{5+5} + \frac{1}{13+5} + \ldots = \frac{5\sqrt{5}}{22};$$

$$\frac{1}{1+1} - \frac{1}{4-1} + \frac{1}{11+1} - \frac{1}{29-1} + \ldots = \frac{\sqrt{5}}{10}.$$

Solution by Professor K. J. Sanjána.

By the usual transformation of series to continued fractions, we get

$$\begin{aligned} \text{(I.)} &= \frac{1}{3} + \frac{1}{15} + \frac{1}{235} + \frac{1}{4183} + \ldots = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.47} + \frac{1}{47.89} + \ldots \\ &= \cfrac{1}{3 - \cfrac{3^2}{3(1+5) - \cfrac{3^2.5^2}{5(3+47) - \cfrac{5^2.47^2}{47(5+89) - \ldots}}}} \\ &\quad \cfrac{1}{10 - \cfrac{1}{2 - \ldots}} = \frac{1}{3 - \frac{1}{2}(10 - 4\sqrt{5})} = \frac{\sqrt{5}+1}{8}; \end{aligned}$$

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$$(\text{II.}) = \frac{1}{4} + \frac{1}{36} + \frac{1}{612} + \frac{1}{10948} + \ldots = \frac{1}{1.4} + \frac{1}{4.9} + \frac{1}{9.68} + \frac{1}{68.161} + \ldots$$

$$= \frac{1}{4-}\,\frac{4^2}{4.10-}\,\frac{4^2.9^2}{9.72-}\,\frac{9^2.68^2}{68.170-}\ldots = \frac{1}{4-}\,\frac{4}{10-}\,\frac{4}{8-}\,\frac{4}{10-}\ldots$$

$$= \frac{1}{4-\frac{1}{10}(40-16\sqrt{5})} = \frac{\sqrt{5}}{8};$$

$$(\text{III.}) = \frac{1}{7} + \frac{1}{91} + \frac{1}{1599} + \frac{1}{28659} + \ldots$$

$$= \frac{1}{1.7} + \frac{1}{7.13} + \frac{1}{13.123} + \frac{1}{123.233} + \ldots$$

$$= \frac{1}{7-}\,\frac{7^2}{7.14-}\,\frac{7^2.13^2}{13.130-}\,\frac{13^2.123^2}{123.246-}\ldots = \frac{1}{7-}\,\frac{1}{2-}\,\frac{1}{10-}\,\frac{1}{2-}\ldots$$

$$= \frac{\sqrt{5}-1}{8}.$$

Similarly, we can prove that

$$\frac{1}{1+5} + \frac{1}{89+5} + \frac{1}{10946+5} + \frac{1}{1346269+5} + \ldots = \frac{1}{6-}\,\frac{9}{25-}\,\frac{1}{5-}\,\frac{1}{25-}\ldots$$

$$= \frac{3\sqrt{5}+5}{66};$$

$$\frac{1}{2+5} + \frac{1}{233+5} + \frac{1}{28657+5} + \ldots = \frac{\sqrt{5}+1}{22} = \frac{3\sqrt{5}+3}{66};$$

$$\frac{1}{5+5} + \frac{1}{610+5} + \frac{1}{75025+5} + \ldots = \frac{3\sqrt{5}}{66};$$

$$\frac{1}{13+5} + \frac{1}{1597+5} + \frac{1}{196418+5} + \ldots = \frac{3\sqrt{5}-3}{66};$$

and

$$\frac{1}{34+5} + \frac{1}{4181+5} + \frac{1}{514229+5} + \ldots = \frac{3\sqrt{5}-5}{66}.$$

Adding up these five results, we get

$$(\text{IV.}) = \frac{15\sqrt{5}}{66} = \frac{5\sqrt{5}}{22}.$$

Lastly, $$(\text{V.}) = \frac{1}{1.2} - \frac{1}{1.3} + \frac{1}{3.4} - \frac{1}{4.7} + \frac{1}{7.11} - \frac{1}{11.18} + \ldots$$

$$= \frac{1}{2.3} + \frac{1}{3.7} + \frac{1}{7.18} + \ldots = \frac{1}{6-}\,\frac{4}{3-}\,\frac{1}{3-}\,\frac{1}{3-}\ldots = \frac{\sqrt{5}}{10}.$$

7092. (R. Russell, B.A.)—Find the locus of a point such that the four normals drawn from it to an ellipse may be harmonic, and show that it passes through the cusps of the evolute, and has them also for cusps.

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Solution by W. J. GREENSTREET, M.A.

The equation $(y-mx)^2(a^2+b^2m^2) = c^4m^2$,

or $b^2x^2.m^4 - 2b^2xy.m^3 + m^2(a^2x^2+b^2y^2-c^4) - 2a^2mxy + a^2y^2 = 0$,

has its roots in H.P. if

$$\begin{vmatrix} b^2x^2, & \frac{1}{2}b^2xy, & \frac{1}{6}(a^2x^2+b^2y^2-c^4) \\ \frac{1}{2}b^2xy, & \frac{1}{6}(a^2x^2+b^2y^2-c^4), & \frac{1}{2}a^2xy \\ \frac{1}{6}(a^2x^2+b^2y^2-c^4), & \frac{1}{2}a^2xy, & a^2y^2 \end{vmatrix} = 0;$$

therefore locus required is

$$a^2b^2x^2y^2\,\tfrac{1}{4}(a^2x^2+b^2y^2-c^4) - \left\{\tfrac{1}{6}(a^2x^2+b^2y^2-c^4)\right\}^3 - \tfrac{1}{4}a^2b^2x^2y^2(a^2x^2+b^2y^2) = 0,$$

or $$\tfrac{1}{4}a^2b^2x^2y^2c^2 = \tfrac{1}{216}\left\{(a^2x^2+b^2y^2-c^4)^3\right\},$$

or $$(a^2x^2+b^2y^2-c^4)^3 + 54a^2b^2x^2y^2c^2 = 0 \quad \ldots\ldots\ldots\ldots\ (1).$$

The cusps of the evolute are $\left\{\pm(a^2-b^2)/a,\ 0\right\}$ and $\left\{0,\ \pm(a^2-b^2)/b\right\}$ which are clearly on this locus.

$$df/dx = 3(a^2x^2+b^2y^2-c^4)^2\,2a^2x + 108a^2b^2xy^2c^2,$$

$$d^2f/dx^2 = 6(a^2x^2+b^2y^2-c^4)\,4a^4x^2 + 6a^2(a^2x^2+b^2y^2-c^4) + 108a^2b^2y^2c^2,$$

$$d^2f/dx\,dy = 6(a^2x^2+b^2y^2-c^4)\,2a^2x.2b^2y + 216a^2b^2xyc^2,$$

$$d^2f/dy^2 = 6(a^2x^2+b^2y^2-c^4)\,4b^4y^2 + 6b^2(a^2x^2+b^2y^2-c^4) + 108a^2b^2x^2c^2;$$

therefore at $\left\{\pm(a^2-b^2)/a,\ 0\right\}$, $d^2f/dx^2\,d^2f/dy^2 = 0 = (d^2f/dx\,dy)^2$;

therefore the Hessian cuts the curve (1) at the cusps of the evolute of the ellipse.

14277. (ROBERT W. D. CHRISTIE.)—Find (1) a general value for x in the equation $3.2^x+1 = pq$ when p is a prime. A few examples are

$x = 3n-2$ (n any integer); then $p = 7$ (*v.* Quest. 14158)
$= 4n-1$,, $p = 5$
$= 10n-3$,, $p = 11$,

and (2) do the same for $3.2^x-1 = pq$.

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.; *and*
H. J. WOODALL, A.R.C.S.

Assuming that p is meant to be a *given* prime, the question is the same as solving the congruences $3.2^x \equiv \mp 1 \pmod{p}$.

(i.) Let x_0 be *any* solution of *either* congruence; then $x = \pm m\xi + x_0$ is the *general* solution of *that* congruence, where ξ is the *least* exponent giving $2^\xi \equiv +1 \pmod{p}$; and, if ξ be even, $x' = \pm m'\xi + \frac{1}{2}\xi + x_0$ is the *general* solution of the other congruence; if ξ be *odd*, the other congruence *has no solution*.

(ii.) As to finding the first solution x_0, let y be a solution of $2^y \equiv 3 \pmod{p}$. Then $2^{y+x_0} = 2^y.2^{x_0} \equiv 3.2^{x_0} \equiv -1$ or $+1 \equiv 2^{\frac{1}{2}\xi}$ or 2^ξ.

$x_0 = \frac{1}{2}\xi - y$ gives a solution of $3.2^{x_0} \equiv -1$, where ξ is *even*.

$x_0 = \xi - y$ gives a solution of $3.2^{x_0} \equiv +1$, whether ξ be *odd* or *even*.

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Or, again, let y' be a solution of $2^{y'} \equiv -3 \pmod{p}$.

Then $2^{y'+x_0} \equiv 2^{y'} . 2^{x_0} \equiv -3 . 2^{x_0} \equiv -1$ or $+1 \equiv 2^{k\xi}$ or 2^{ξ};

$x'_0 = \frac{1}{2}\xi - y'$ gives a solution of $3 . 2^{x} \equiv +1$, when ξ is *even*,

$x'_0 = \xi - y'$ gives a solution of $3 . 2^{x} \equiv -1$, when ξ is *odd* or *even*.

[If any of the above values of x, x', x_0, x'_0 be negative, they should be rendered positive by the addition of multiples of ξ.]

(iii.) As to the *possibility* of either $2^{y} \equiv +3$, $2^{y'} \equiv -3 \pmod{p}$:—Let ξ, η, η' be the *least* exponents giving $2^{\xi} \equiv 3^{\eta} \equiv (-3)^{\eta'} \equiv +1 \pmod{p}$. Let G be the G.C.M. of ξ, y, so that $\xi = \alpha G$, $y = \beta G$, where α, β are *mutually prime*. Then

$$3^{\alpha} \equiv (2^{y})^{\alpha} \equiv (2^{\beta G})^{\alpha} = (2^{\alpha G})^{\beta} \equiv (2)^{\beta} \equiv +1 \equiv (3^{\eta})^{k};$$

therefore $\alpha = k\eta$ and $\xi = \alpha G = kG . \eta$, whence $\xi \div \eta = integer$.

Similarly, if $2^{y'} \equiv -3$, then $\xi \div \eta' = integer$.

Thus $\xi \div \eta$ or $\xi \div \eta' = integer$ is the condition that $2^{y} \equiv +3$ or $2^{y'} \equiv -3$. Hence the following scheme :—

Data.		*Results.*
$\frac{\xi}{\eta} = integer,$	η and ξ *odd* ;	$2^{y} \equiv +3$, but not $2^{y'} \equiv -3$,
	η *even* ;	$2^{y} \equiv +3$, and $2^{y'} \equiv -3$.
$\frac{\xi}{\eta'} = integer,$	η' and ξ *odd* ;	$2^{y'} \equiv -3$, but not $2^{y} \equiv +3$,
	η' *even* ;	$2^{y'} \equiv -3$, and $2^{y} \equiv +3$.

(iv.) The properties of 2-ic residues show that the following are the possible *linear* forms of *prime* divisors (p) of $(2^{y} \mp 3)$,

$$(2^{y} - 3) \begin{cases} p = 24w + 1, 5, 19, 23, \text{ when } y \text{ is } odd, \\ p = 24w + 1, 11, 13, 23, \text{ when } y \text{ is } even ; \end{cases}$$

$$(2^{y} + 3) \begin{cases} p = 24w + 1, 5, 7, 11, \text{ when } y \text{ is } odd, \\ p = 24w + 1, 7, 13, 19, \text{ when } y \text{ is } even. \end{cases}$$

Thus, primes $p = 24w + 7$, 17 are non-divisors of $(2^{y} - 3)$,

$p = 24w + 17$, 23 are non-divisors of $(2^{y} + 3)$.

(v.) The actual solution (y) of $2^{y} \equiv \pm 3 \pmod{p}$ can be picked out at sight from JACOBI'S *Canon Arithmeticus* for *all* primes (in the possible cases) < 1000. For primes > 1000 the solution can (as far as known to the author) only be done by a rather laborious tentative process.

The SOLVER has prepared a MS. Table of the solutions (y) for all primes < 2300, and for a good many higher ones, by aid of which it has been possible to factorize $(2^{y} \mp 3)$ and $(3 . 2^{x} \pm 1)$ to pretty high numbers.

[The PROPOSER adds the following :—

(A) In the theorem 14250, if we assume $x = 1$, $k = 1$, $a = 2$, n any integer, $(XP + 1)/2 = 3 . 2^{z}$, we easily obtain $3 . 2^{np+z+1} \equiv 1 \pmod{P}$ when $P = 5, 11, 23, \ldots 6q - 1$, p being obtained from the equation $a^{p} \equiv 1 \pmod{P}$.

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To obtain general values for a given prime, we may proceed as follows:—

(1) Take $P = 5$; then $2^p \equiv 1 \pmod 5$, which gives $p = 4m$ (m any integer). Hence $3.2^{4mn+1} \equiv 1 \pmod 5$ for all values of m and n.

(2) $P = 23$, $p = 11$, $z = 2$. Hence $3.2^{11n+3} \equiv 1 \pmod{23}$.

Other examples are $3.2^{10n+2} \equiv 1 \pmod{11}$, $3.2^{23n+4} \equiv 1 \pmod{47}$.

(B) $X = 5$ gives $P = 19, 307, 4915\ (983) \ldots (3.2^{4m+1}-1)/5$,

$z = 4m$ for all values of m,

$p = 9$ from $2^p \equiv 1 \pmod{19}$.

Thus $3.2^{9n+5} \equiv \pm 1 \pmod{19}$.

Other examples are $X = 11$, $P = 1117, 1143901$, &c.; $z = 11 \ldots 10m-9$.

Thus $3.2^{np+12} \equiv 1 \pmod{1117}$ and $3.2^{np+22} \equiv 1 \pmod{1143901}$,

$X = 23$, $P = 2137, 4376665\ (875333)$, $z = 11n-9$.

Thus $3.2^{np+14} \equiv 1 \pmod{2137}$.

(C) $x = 1$, $a = 2$, $k = 1$, but, making $(XP-1)/2 = 3.2^x$, we easily establish $3.2^{pn+x}+1 \equiv 0 \pmod p$.

Some examples are

$P = 7$, $z = 1$, $p = 3$; then $3.2^{3n+1}+1 = 7Q$.

$P = 13$, $z = 2$, $p = 12$; then $3.2^{12n+2}+1 = 13Q$.

$P = 97$, $z = 5$, $p = 48$; then $3.2^{48n+5}+1 = 97Q$, &c.

(D) By the help of a generalized form of FERMAT's theorem it is not difficult to show that, if $3.2^x+1 = pq$, then

$$3.2^{(p-1)(n-1)+x}+1 = pq \text{ also } (n \text{ any integer}).$$

E.g.:

$x = 3$, $p = 5$;	then 3.2^{4n-1}	$+1 = 5q$,	also 3.2^{4n-3}	$-1 = 5q$.
$x = 4$, $p = 7$;	,, 3.2^{6n-2}	$+1 = 7q$.		
$x = 7$, $p = 11$;	,, 3.2^{10n-3}	$+1 = 11q$,	,, 3.2^{10n-8}	$-1 = 11q$.
$x = 2$, $p = 13$;	,, 3.2^{12n-10}	$+1 = 13q$,	,, 3.2^{12n-4}	$-1 = 13q$.
$x = 14$, $p = 19$;	,, 3.2^{18n-4}	$+1 = 19q$,	,, 3.2^{18n-13}	$-1 = 19q$.
$x = 9$, $p = 29$;	,, 3.2^{28n-19}	$+1 = 29q$,	,, 3.2^{28n-5}	$-1 = 29q$.
$x = 9$, $p = 53$;	,, 3.2^{52n-43}	$+1 = 53q$,	&c., &c.	
$x = 5$, $p = 97$;	,, 3.2^{96n-91}	$+1 = 97q$.		
$x = 14$, $p = 199$;	,, $3.2^{198(n-1)+14}$	$+1 = 199q$.		

And generally, if $2^p = PN+1$, where p = period or $P-1$, and P is a factor of 3.2^x+1, a theorem can be formed similar to the above.

Incidentally it may be noticed that $3.2^x+1 = pq$ ends only in 3, 5, 7, or 9.]

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12186. (Professor ZERR.)—Venus transits the Sun's disc at such a rate that the Sun's diameter S would be traversed in 7¼ hours. At the same time it is observed that the Sun in his annual course moves through a distance equal to his own apparent diameter in 12 hours; assuming circular orbits and uniform motion, determine the Earth's distance from the Sun, the distance of Venus being unity.

Solution by Professors ZERR, KRISHMANACHARRY, *and others.*

Let t = time for Venus; t' = time for Sun to go through a distance equal to the Sun's diameter; and ρ = distance from Earth to Sun; then, if ω be the angular velocity, we have,

$$\omega\rho/(1+\sqrt{\rho})+\omega=\omega t'/t;\quad \text{therefore}\quad t\rho=(t'-t)+(t'-t)\sqrt{\rho};$$

$$\text{therefore}\quad \rho=\left\{(t'-t)/2t^2\right\}\left\{(t'+t)+\sqrt{(t'^2-3t^2+2tt')}\right\}$$

$$=\tfrac{2}{4}\tfrac{1}{8}\tfrac{1}{4}\left(\tfrac{2}{3}\tfrac{8}{3}+\tfrac{1}{3}\sqrt{1428}\right)=1{\cdot}38538.$$

14590. (G. D. WILSON, B.A.) — If the conic represented by $lx^2+my^2+nz^2=0$ (areal coordinates) be a parabola, show that the equation of its axis is

$$\frac{\cos A}{al}(my-nz)+\frac{\cos B}{bm}(nz-lx)+\frac{\cos C}{cn}(lx-my)=0.$$

Solution by Professor K. J. SANJÁNA, M.A.

Putting $a\alpha$, $b\beta$, $c\gamma$ for x, y, z, respectively, the trilinear equation of the conic becomes $\Sigma la^2\alpha^2=0$. The condition that this should represent a parabola (*i.e.*, touch the line at infinity) gives $mn+nl+lm=0$. The coordinates (f, g, h) of the focus are given by

$$mnb^3c^3af-mnb^2c^2\Delta=nlc^2a^2bg-nlc^3a^2\Delta=lma^2b^2ch-lma^2b^3\Delta$$

and $af+bg+ch=2\Delta$ (SALMON, § 384, Ex. 3); whence we get

$$f:g:h=(mb^2+nc^2)/a:(nc^2+la^2)/b:(la^2+mb^2)/c.$$

The directrix, being the polar of the focus, is given by

$$al\alpha(mb^2+nc^2)+bm\beta(nc^2+la^2)+cn\gamma(la^2+mb^2)=0.$$

Let the equation of the axis be $L\alpha+M\beta+N\gamma=0$. Since the axis passes through the focus, we have

$$L(mb^2+nc^2)/a+M(nc^2+la^2)/b+N(la^2+mb^2)/c=0\ldots\ldots\ldots\ldots(1).$$

Also, since the axis is perpendicular to the directrix, we have

$$\Sigma L\left\{al(mb^2+nc^2)-cn(la^2+mb^2)\cos B-bm(nc^2+la^2)\cos C\right\}=0;$$

$$\text{therefore}\quad \Sigma L\left\{nlcab\cos A+lmabc\cos A-mnbca\cos(B-C)\right\}=0;$$

$$\text{therefore}\quad \Sigma L\left\{(nl+ln)\cos A-mn\cos(B-C)\right\}=0;$$

$$\text{therefore}\quad \Sigma L\left\{mn\cos A+mn\cos(B-C)\right\}=0,\quad \text{for}\quad \Sigma mn,=0;$$

$$\text{therefore}\quad \Sigma Lmn\sin B\sin C=0\quad\text{or}\quad Lmn/a+Mnl/b+Nlm/c=0\ldots\ldots(2).$$

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From (1) and (2) we get

$$L \propto l(mnc^2 + mla^2 - nla^2 - mnb^2)/bc \propto al\{mla^2 - nla^2 + (b^2 - c^2)(ml + nl)\},$$

for $\Sigma mn = 0$; thus

$$L \propto al^2(2mab\cos C - 2nca\cos B) \propto a^2l^2(mb\cos C - nc\cos B).$$

Hence the axis is $\Sigma a^2l^2a(mb\cos C - nc\cos B) = 0$.

Put x for aa, &c., and divide by lmn; thus

$$\Sigma ax\left(\frac{lb\cos C}{n} - \frac{lc\cos B}{m}\right) = 0,$$

or

$$\frac{\cos A}{l}bc(my - nz) + \frac{\cos B}{m}ca(nz - lx) + \frac{\cos C}{n}ab(lx - my) = 0.$$

14488. (Professor Umes Chandra Ghosh.)—If the vertical angle A and base BC ($= a$) of a triangle ABC are given, show that the locus of its Brocard points is

$$(x^2 + y^2)^2 - a(x^2 + y^2)(x + y\cot A) + 4R^2y^2 = 0,$$

R being the radius of the circumcircle of the triangle, B the origin, and BC the axis of x.

Solution by W. J. Greenstreet, M.A.

For ω we have $y = 2R\sin^2\omega\, c/b$ $= 2R\sin^2\omega\, c/b\cot\omega$,

$$(x^2 + y^2)^2 + a(x + y\cot A)(x^2 + y^2) + 4R^2y^2$$
$$= 8R^3\sin^3\omega\, c^2/b^4\{2R\sin\omega(c^2 + b^2) - abc(\cos\omega + \sin\omega\cot A)\}$$

This vanishes if $2R\tan\omega(c^2 + b^2) = abc + abc\tan\omega\cot A$;

if $R(a^2 + b^2 + c^2) = abc\tan\omega$, because $abc\cot A = R(b^2 + c^2 - a^2)$,

if $\Sigma a^2 = 4\Delta\cot\omega$, which is the case; therefore the locus for ω is as stated. Similarly for ω'.

5812. (R. A. Roberts, M.A.)—Prove that the locus of the centroid of a triangle inscribed in one conic and circumscribed about another is a conic.

Note by W. J. Greenstreet, M.A.

Mr. Roberts (*Collection of Examples on the Analytical Geometry of Plane Conics*, Longmans, 1884, p. 7) gives an elegant proof, finding as the locus

$$9C(x^2/a^2 + y^2/b^2) - 12(Gx/a^2 + Fy/b^2) + 2(A/a^2 + B/b^2) + C' = 0,$$

where $U \equiv x^2/a^2 + y^2/b^2 - 1 = 0$ is the circum- and

$$\Sigma \equiv (A, B, C, F, G, H)(\lambda, \mu, \nu)^2 = 0$$

the in-conic.

The centroid is fixed if $A/a^2 = B/b^2$, $H = 0$.

If U is a parabola, $C = 0$, and the locus is a straight line.

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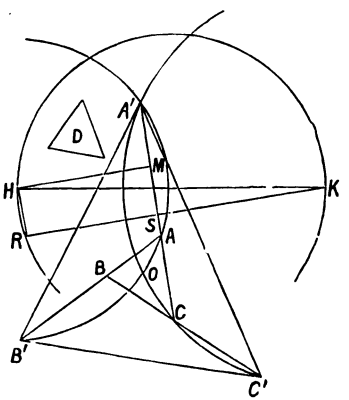
14647. (Salutation.)—ABC being any plane triangle, produce CA to A′, AB to B′, and BC to C′, so that A′B′C′ may be similar to another given triangle, and m times as large as ABC.

Solution by J. H. Taylor, M.A.

Analysis.—Let A′B′C′ be a triangle similar to the given triangle D, and m times the given triangle ABC. (Euclid, Book vi., Propositions 19 and 25). On A′B′ describe arc containing angle $= \angle B + \angle C$,
on A′C′ $= \angle A + \angle B$,
on B′C′ $= \angle C + \angle A$.



On these arcs, which intersect in O, the angles A, B, C must lie. Let H, K be the centres of the circles A′OB′, A′OC′. On HK as diameter describe a circle, and in this place a chord HR $= \frac{1}{2}$AC ; through A′ draw A′AC parallel to HR, cutting the circles A′OB′, A′OC′ in A, C respectively ; from AB′ cut off AB of the given length ; then ABC is the given triangle.

Proof.—Join KR, cutting A′AC in S ; then $\angle$ KRH is a right angle, and therefore KSA′ is a right angle. Draw HM perpendicular on A′A; then

$$A'M = MA, \quad A'S = SC, \quad CS + SA = A'S + SA = A'M + MA;$$

whence $$AC = 2MS = 2HR.$$

The synthesis is obvious.

The problem on which this proof depends is given on Book iii. in Potts' *Euclid* (1845), p. 316, No. 38 ; but in *Hints*, p. 40, without giving the solution, he says : "The proof of this problem involves proportion."

14653. (Professor E. J. Nanson.)—If

$$a = (x^2 - y)/(1 - xy), \quad b = (y^2 - x)/(1 - xy),$$

prove that $$(a^2 - b)/(1 - ab) = x, \quad (b^2 - a)/(1 - ab) = y.$$

Solution by H. J. Woodall, A.R.C.S. ; James Blaikie, M.A. ; *and many others.*

Substituting, we get $(a^2 - b)/x = (b^2 - a)/y = (x^3 + y^3 - 3xy + 1)/(1 - xy)^2$, and $1 - ab$ is similarly shown to be equal to this same expression ; whence the theorem.

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6723. (C. Leudesdorf, M.A.)—A pair of tangents to a given conic form an harmonic pencil with two straight lines whose directions are given and which include a right angle. Show that the locus of the point of intersection of the tangents is a rectangular hyperbola, except in the case where the given conic is a parabola, when the locus is a straight line.

Solution by A. M. Nesbitt, M.A.

Refer the conic to central axes parallel to the two given perpendicular straight lines. Let its equation be $ax^2+2bxy+by^2+1=0$. Then the condition that $y-\beta=m(x-\alpha)$ shall touch the conic is

$$0=\begin{vmatrix} a, & h, & 0, & m \\ h, & b, & 0, & -1 \\ 0, & 0, & 1, & \beta-m\alpha \\ m, & -1, & \beta-m\alpha, & 0 \end{vmatrix}$$

or $$m^2\{a^2(ab-h^2)+b\}+2m\{h-a\beta(ab-h^2)\}+(ab-h^2)\beta^2+a=0,$$

and by conditions of question we have to eliminate m, between this and the equation got by changing sign of m. Hence $h=a\beta(ab-h^2)$, which is an equilateral hyperbola.

In the case of the non-central conic, take $y^2=4ax$ for the parabola, $y=mx+a/m$, $y=nx+a/n$ for the tangents; then the conditions of the question give us

$$(m+n)/(1-mn)=\text{const.}=\lambda \quad \text{and} \quad mn=a/x,\ (m-n)y=x(m^2-n^2);$$

whence $$\lambda=(y/x)/(1-a/x) \quad \text{or} \quad y=\lambda(x-a),$$

a straight line through focus.

14704. (Professor G. B. Mathews, F.R.S.)—If x, y independently assume the values $0, \pm1, \pm2, \pm3$, &c., prove that

$$\Sigma e^{-\pi(x^2+2y^2)}=\frac{\sqrt{(4+2\sqrt{2})}}{\pi}\int_0^1\frac{dx}{\sqrt{(1-x^4)}}.$$

Solution by the Proposer.

In Jacobi's notation $$\Sigma q^{x^2}=\sqrt{(2K/\pi)},$$

and, changing q into q^2, K becomes $\frac{1}{2}(1+\kappa')K$, so that

$$\Sigma q^{x^2+2y^2}=\sqrt{\{\tfrac{1}{2}(1+\kappa')\}}\,2K/\pi,$$

Now, when $$\kappa=\kappa'=\tfrac{1}{2}\sqrt{2}, \quad K=\sqrt{2}\int_0^1\frac{dx}{\sqrt{(1-x^4)}},$$

and $q=e^{-\pi}$; hence $$\Sigma e^{-\pi(x^2+2y^2)}=\sqrt{\left(\frac{2+\sqrt{2}}{4}\right)}\frac{2\sqrt{2}}{\pi}\int_0^1\frac{dx}{\sqrt{(1-x^4)}}=\text{\&c.}$$

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14606. (R. Tucker, M.A.)—ABC, A′B′C′ are the in- and circum-triangles of the same circle, and their sides are parallel. Prove

$$AB = 4\Pi (\sin \tfrac{1}{2}A)\, A'B',$$

and that A′A, B′B, C′C meet in

$$(s-a)\,\alpha/a = (s-b)\,\beta/b = (s-c)\,\gamma/c.$$

Solution by R. P. Paranjpye, B.A.; Rev. T. Roach, M.A.; *and others.*

Let ρ be the radius of the circle; then, from the triangle ABC,

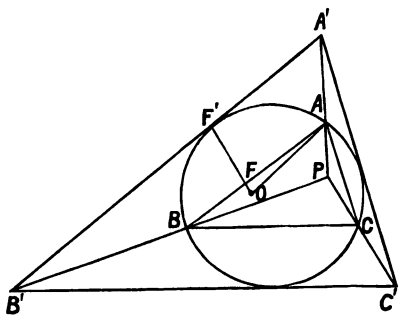
$$AB = 2\rho \sin C,$$

and, from triangle A′B′C′,

$$A'B' = \rho(\cot \tfrac{1}{2}A + \cot \tfrac{1}{2}B);$$

therefore

$$\frac{AB}{A'B'} = \frac{2 \sin C}{\cot \tfrac{1}{2}A + \cot \tfrac{1}{2}B} = 4\Pi (\sin \tfrac{1}{2}A).$$



Again, taking ABC as the triangle of reference for trilinear coordinates, we have, since

$$FF' = \rho(1-\cos C),$$

therefore, for A′B′, $\gamma = -\rho(1-\cos C)$; therefore the equation of A′B′ is

$$\gamma\Delta + \sin^2 \tfrac{1}{2}C \,.\, \rho(a\alpha + b\beta + c\gamma) = 0.$$

Similarly the equation of A′C′ is

$$\beta\Delta + \sin^2 \tfrac{1}{2}B \,.\, \rho(a\alpha + b\beta + c\gamma) = 0;$$

therefore the equation of AA′ is $\gamma/\sin^2 \tfrac{1}{2}C = \beta/\sin^2 \tfrac{1}{2}B$; therefore P is the point

$$\alpha/\sin^2 \tfrac{1}{2}A = \beta/\sin^2 \tfrac{1}{2}B = \gamma/\sin^2 \tfrac{1}{2}C,$$

which is the same as that given in the enunciation.

6405 & 6531. (The late Professor Sylvester, F.R.S.)—(6405.) If p, q, r, s are the distances of a point in a circular cubic from the four concyclic foci A, B, C, D, prove that

$$\frac{(p-q)(q-r)(r-p)}{ABC} = \frac{(q-r)(r-s)(s-q)}{BCD} = \frac{(r-s)(s-p)(p-r)}{CDA} = \frac{(s-p)(p-q)(q-s)}{DAB}.$$

(6531.) Prove that only one proper circular cubic can be drawn having four concyclic foci at the angles of a trapezoid. (May be solved with Quest. 6405, reproposed in *Ed. Times* of March, 1900.)

[For another solution of Quest. 6405, see *Reprint*, Vol. lxxiii., p. 121.]

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Remarks by H. M. Taylor, M.A., F.R.S.

(1) Two circular cubics can be described having the same four concyclic points A, B, C, D for foci.

(2) These curves intersect in nine points: *i.e.*, O, P, Q, the three centres of the quadrangle A, B, C, D; L, M, N, the feet of the perpendiculars, and R, the orthocentre, of the triangle OPQ; and I, J, the circular points at infinity. The point R is the centre of the circle ABDC.

(3) The two cubics cut at right angles at each of the nine points I, J, L, M, N, O, P, Q, and R.

(4) Each of the two cubics will invert into itself with respect to any of the four points O, P, Q, R as pole of inversion.

(5) If, as is the case in the diagram (Fig. 1), the point P be within the circle ABCD, one of the cubics consists of an infinite branch OLR (Fig. 2) and an oval PNQM, and the other cubic consists of an infinite branch QNR (Fig. 3) and an oval OM'L.

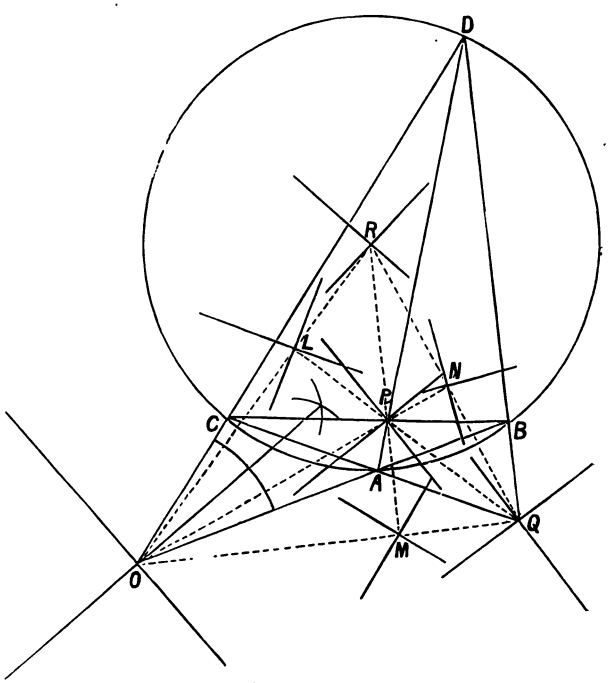

Fig. 1.

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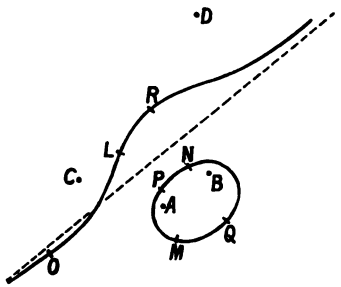

Fig. 2.

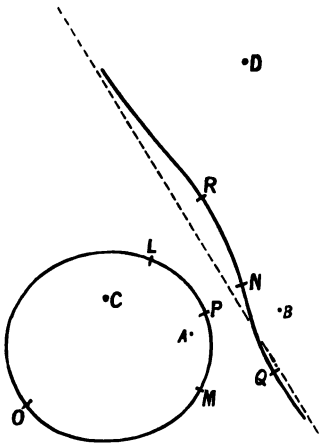

Fig. 3.

(6) In the first curve, the infinite branch OLR inverts into itself, and the oval PNQM into itself, when either O or R is the pole of inversion; but the infinite branch inverts into the oval, and *vice versa*, when either P or Q is the pole. A similar remark applies to the second curve.

(7) If ρ_1, ρ_2, ρ_3, ρ_4 represent the distances of any point S on the curve [and a, b, c, d the distances of the point O] from the points A, B, C, D, respectively, then the infinite branch OLR of the first curve is represented by any of the four equations—

$$\begin{aligned}
(c-d)\,\rho_2 + (d-b)\,\rho_3 + (b-c)\,\rho_4 &= 0,\\
(c-d)\,\rho_1 + (d-a)\,\rho_3 + (a-c)\,\rho_4 &= 0,\\
(b-d)\,\rho_1 + (d-a)\,\rho_2 + (a-b)\,\rho_4 &= 0,\\
(b-c)\,\rho_1 + (c-a)\,\rho_2 + (a-b)\,\rho_3 &= 0;
\end{aligned}$$

and the oval PNQM by any of the equations—

$$\begin{aligned}
(c-d)\,\rho_2 - (d-b)\,\rho_3 + (b-c)\,\rho_4 &= 0,\\
(c-d)\,\rho_1 + (d-a)\,\rho_3 - (a-c)\,\rho_4 &= 0,\\
-(b-d)\,\rho_1 + (d-a)\,\rho_2 + (a-b)\,\rho_4 &= 0,\\
(b-c)\,\rho_1 - (c-a)\,\rho_2 + (a-b)\,\rho_3 &= 0.
\end{aligned}$$

Similarly, with regard to the other curve, the infinite branch QNR is represented by any of the equations—

$$\begin{aligned}
-(c-d)\,\rho_2 - (d+b)\,\rho_3 + (b+c)\,\rho_4 &= 0,\\
-(c-d)\,\rho_1 - (d+a)\,\rho_3 + (a+c)\,\rho_4 &= 0,\\
(b+d)\,\rho_1 - (d+a)\,\rho_2 + (a-b)\,\rho_4 &= 0,\\
(b+c)\,\rho_1 - (c+a)\,\rho_2 + (a-b)\,\rho_3 &= 0;
\end{aligned}$$

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and the oval OMPL by any of the equations—

$$\begin{aligned}
&(c-d)\,\rho_2-(d+b)\,\rho_3+(b+c)\,\rho_4=0,\\
(c-d)\,\rho_1 &\quad -(d+a)\,\rho_3+(a+c)\,\rho_4=0,\\
(b+d)\,\rho_1-(d+a)\,\rho_2 &\quad -(a-b)\,\rho_4=0,\\
(b+c)\,\rho_1-(c+a)\,\rho_2-(a-b)\,\rho_3 &\quad =0.
\end{aligned}$$

8. From the equations which are true for the infinite branch of either curve, we can obtain equations corresponding to those of Quest. 6405; but from the equations which represent an oval we obtain equations of a different form. For instance, from the equations of the oval PNQM, we obtain

$$\frac{(\rho_3+\rho_3)(\rho_3+\rho_4)(\rho_4-\rho_2)}{\triangle BCD}=\frac{(\rho_3+\rho_4)(\rho_4+\rho_1)(\rho_1-\rho_3)}{\triangle ACD}=\frac{(\rho_4+\rho_1)(\rho_1+\rho_2)(\rho_2-\rho_4)}{-\triangle ABD}$$

$$=\frac{(\rho_1+\rho_2)(\rho_2+\rho_3)(\rho_3-\rho_1)}{-\triangle ABC}.$$

9. If the quadrangle be a trapezoid, then $a=c$, $b=d$. In this case, all the equations which represent the infinite branch OLR and the oval PNQM of the first curve degenerate, and each degenerate equation represents only the straight line OLR.

10. The exact nature of the degeneration is perhaps better seen from the Cartesian equation of the curves. If OAB, OCD be taken for the axis of x and y respectively, and ω be the angle between the axes, the two curves are represented by the equations—

$$\begin{aligned}
&2(a+b\mp c\mp d)(x^2+2xy\cos\omega+y^2)(x\mp y)\\
&-\{(a+b)^2\mp 2(a+b)(c+d)+4ab\pm(a-b)^2\cos\omega\}x^2\\
&-2\left[\{(a+b)(c+d)\mp 4ab\}(1\mp\cos\omega)\right]xy\\
&-\{(c+d)^2\mp 2(a+b)(c+d)+4ab\pm(c-d)^2\cos\omega\}y^2\\
&+2(a+b\mp c\mp d)ab(x\mp y)=0,
\end{aligned}$$

the upper or the lower sign being taken throughout.

If ABCD be a trapezoid, so that $a=c$ and $b=d$, the equation with the upper sign in this case reduces to

$$(a-b)^2(1-\cos\omega)(x-y)^2=0.$$

This shows that one of the confocal circular cubics reduces to the line $x=y$ counted twice and the line at infinity.

On the subject of circular cubics, reference may be made to SALMON'S *Higher Plane Curves* (3rd edition, pp. 247–250).

[The circle ABDC in each of the three figures represents one and the same circle.]

6704. (The late Professor WOLSTENHOLME, M.A.)—A circle is drawn touching the parabola $2a=r(1+\cos\theta)$, passing through the focus S, and meeting the parabola again in two points; prove that (1) the locus of T, the common point of the two common tangents to the circle and parabola, is $a=4r(\cos\frac{1}{2}\theta)^4$, the second negative pedal of the parabola $a=2r(1+\cos\theta)$; (2) the tangent at T to the locus of T bisects at right angles the radius of

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the circle drawn from S; and (3) the locus of the centre of the circle is the first negative pedal of the parabola $a = r(1+\cos\theta)$.

Solution by Rev. G. Richardson, M.A.

Let α be the angle vector of P where the circle touches parabola; O the centre of circle, OP = OS = C. Then $\angle OSP = OPS = \frac{1}{2}\alpha$; therefore $APO = \alpha + \frac{1}{2}\alpha = \frac{3}{2}\alpha$. Also $2C\cos\frac{1}{2}\alpha = SP = a/\cos^2\frac{1}{2}\alpha$, and equation of circle is
$$a\cos(\theta - \tfrac{3}{2}\alpha) = r\cos^3\tfrac{1}{2}\alpha \quad \ldots\ldots(1).$$
Tangent to parabola at (β) is
$$2a/r = \cos\theta + \cos(\theta-\beta) \quad \ldots\ldots(2),$$
tangent to circle at (γ) is
$$2a/r = 2\cos(\theta + \tfrac{3}{2}\alpha - 2\gamma)\cos^3\tfrac{1}{2}\alpha/\cos^2(\tfrac{3}{2}\alpha - \gamma) \quad \ldots\ldots(3).$$
If (2) and (3) are identical,
$$1+\cos\beta = 2\cos(2\gamma - \tfrac{3}{2}\alpha)\cos^3\tfrac{1}{2}\alpha/\cos^2(\tfrac{3}{2}\alpha - \gamma) \quad \ldots\ldots(4)$$
and
$$\sin\beta = 2\sin(2\gamma - \tfrac{3}{2}\alpha)\cos\tfrac{1}{2}\alpha/\cos^2(\tfrac{3}{2}\alpha - \gamma) \quad \ldots\ldots(5).$$
(5)/(4) gives $\tan\frac{1}{2}\beta = \tan(2\gamma - \frac{3}{2}\alpha)$; therefore $4\gamma = 3\alpha + \beta$;
(4) becomes $\cos\frac{1}{2}\beta\cos^2(\frac{3}{4}\alpha - \frac{1}{4}\beta) = \cos^3\frac{1}{2}\alpha$;
therefore $2\cos\frac{1}{2}\beta + \cos(\frac{3}{2}\alpha - \beta) + \cos\frac{3}{2}\alpha = 4\cos^3\frac{1}{2}\alpha = \cos\frac{3}{2}\alpha + 3\cos\frac{1}{2}\alpha$;
therefore $2\sin\frac{1}{4}(\alpha-\beta)\sin\frac{1}{4}(\alpha+\beta) - \sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(2\alpha-\beta) = 0$;
therefore $\sin\frac{1}{4}(\alpha-\beta) = 0$,
or $2\sin\frac{1}{4}(\alpha+\beta) = 2\cos\frac{1}{4}(\alpha-\beta)\sin\frac{1}{2}(2\alpha-\beta) = \sin\frac{1}{4}(5\alpha-3\beta) + \sin\frac{1}{4}$ —
therefore $\sin\frac{1}{2}(\alpha-\beta)\cos\frac{1}{4}(3\alpha-\beta) + \sin\frac{1}{4}(\alpha-\beta)\cos\frac{1}{2}\alpha = 0$;
therefore $\sin\frac{1}{4}(\alpha-\beta) = 0$ or $\cos\frac{1}{2}(2\alpha-\beta) + 2\cos\frac{1}{2}\alpha = 0$;
therefore $(\cos\alpha\cos\frac{1}{2}\beta)^2 = (2\cos\frac{1}{2}\alpha + \sin\alpha\sin\frac{1}{2}\beta)^2$
or $\sin^2\frac{1}{2}\beta + 4\cos\frac{1}{2}\alpha\sin\alpha\sin\frac{1}{2}\beta + 4\cos^2\frac{1}{2}\alpha - \cos^2\alpha = 0$.
If β_1, β_2 are the roots of the equation, then
$$\sin\tfrac{1}{2}\beta_1\sin\tfrac{1}{2}\beta_2 = 4\cos^2\tfrac{1}{2}\alpha - \cos^2\alpha.$$
Similarly $\cos\frac{1}{2}\beta_1\cos\frac{1}{2}\beta_2 = 4\cos^2\frac{1}{2}\alpha - \sin^2\alpha$;
therefore $\cos\frac{1}{2}(\beta_1+\beta_2) = \cos^2\alpha - \sin^2\alpha = \cos 2\alpha$ or $\beta_1+\beta_2 = 4\alpha$,
and the locus required is found by eliminating α, β_1, β_2, and from
$$2a/r - \cos\theta = \cos(\theta-\beta_1) = \cos(\theta-\beta_2) \quad [\textit{vide}\ (2)].$$
From these latter
$$2\theta = \beta_1+\beta_2 = 4\alpha \text{ and } 2a/r - \cos\theta = \cos\left\{\theta - \tfrac{1}{2}(\beta_1+\beta_2)\right\}\cos\tfrac{1}{2}(\beta_1-\beta_2),$$
or
$$2a/r = \cos\theta + (4\cos^2\tfrac{1}{2}\alpha - \sin^2\alpha) + (4\cos^2\tfrac{1}{2}\alpha - \cos^2\alpha)$$
$$= \cos\theta + 8\cos^2\tfrac{1}{2}\theta - 1 = 8\cos^2\tfrac{1}{2}\theta - 2\sin^2\tfrac{1}{2}\theta;$$
therefore $a/r = 4\cos^4\frac{1}{4}\theta$, which proves (1).

(2) The tangent to this locus at θ' is $a/r = 4\cos^3\frac{1}{4}\theta'\cos(\theta - \frac{3}{4}\theta')$, which may be written $2c\cos^3\frac{1}{4}\alpha = 4r\cos^3\frac{1}{4}\alpha\cos(\theta - \frac{3}{4}\alpha)$, which obviously bisects SO at right angles.

(3) The locus of the centre of the circle is found from $a/2c = \cos^3\frac{1}{4}\alpha$, which gives $a/2r = \cos^3\frac{1}{3}\theta$, the first negative pedal of $a = r(1+\cos\theta)$.

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14576. (Professor Cochez.)—Courbe $(x^2-y^2)xy = 1$.

Solution by v. Ragunath Rau, B.A.

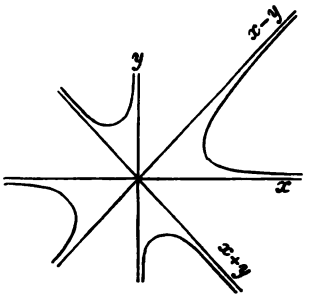
Changing to polar coordinates, the equation becomes

$(r^2\cos^2\theta - r^2\sin^2\theta)\, r^2 \sin\theta\cos\theta = 1,$

or $r^2\cos 2\theta \,.\, r^2 \tfrac{1}{2}(\sin 2\theta) = 1,$

or $r^2 \tfrac{1}{4}(\sin 4\theta) = 1,$

or $r^2 \sin 4\theta = 4.$



Since r has two equal values of opposite sign, the curve is central, the origin being the centre. This is also evident from the consideration that the original equation remains unaltered if x and y are changed to $-x$ and $-y$. It is also evident that the lines $x-y$ and $x+y$ and the coordinate axes are asymptotes. For values of θ from 0 to $\frac{1}{4}\pi$, r is real, becoming infinite for the extreme values. For values of θ from $\frac{1}{4}\pi$ to $\frac{1}{2}\pi$, r is unreal, so no part of the curve lies between $y-$ axis and the line $x-y$. For values of θ from $\frac{1}{2}\pi$ to $\frac{3}{4}\pi$, r is again real, and unreal as θ ranges from $\frac{3}{4}\pi$ to π; so the curve is as traced in the figure.

14675. (The late Professor Wolstenholme, M.A., Sc.D.)—In the tetrahedron OABC the sum of the faces ABC, OBC is equal to the sum of the faces OCA, OAB. Prove that

$$(a-x)/\sin\tfrac{1}{2}(X-A) = (b-y)/\sin\tfrac{1}{2}(B-Y) = (c-z)/\sin\tfrac{1}{2}(C-Z)$$
$$= \text{some positive quantity.}$$

[The lengths OA, OB, OC are denoted by a, b, c; and the lengths BC, CA, AB by x, y, z; and the dihedral angles opposite these edges by A, B, C, X, Y, Z respectively.]

Solution by H. W. Curjel, M.A.

From Quest. 13521 (Vol. lxviii., p. 102)

$$\left\{\frac{a-x}{\sin\frac{1}{2}(X-A)}\right\}^2 = \left\{\frac{b-y}{\sin\frac{1}{2}(B-Y)}\right\}^2 = \ldots = \left\{\frac{a+x}{\sin\frac{1}{2}(A+X)}\right\}^2$$
$$= \ldots = 4xa/(\sin X \sin A) = \ldots,$$

for $\Delta_0+\Delta_1 = \Delta_2+\Delta_3$; whence $u' = 0$.

If r is the in-radius,

$$rx\cot\tfrac{1}{2}A = ra\cot\tfrac{1}{2}X \quad \text{for} \quad \Delta_0+\Delta_1 = \Delta_2+\Delta_3;$$

therefore $\dfrac{a-x}{\sin\frac{1}{2}(X-A)}$ is positive, since A and X are both less than two right angles.

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If $\Delta_1 \neq \Delta_2$, then $rb \cot \frac{1}{2}Y \neq ry \cot \frac{1}{2}B$; therefore

$$b/(\sin \tfrac{1}{2}Y \cos \tfrac{1}{2}A) \neq y/(\sin \tfrac{1}{2}B \cos \tfrac{1}{2}Y);$$

but $(b+y)/(\sin \frac{1}{2}Y \cos \frac{1}{2}B + \cos \frac{1}{2}Y \sin \frac{1}{2}B)$

$$= \surd[b/(\sin \tfrac{1}{2}Y \cos \tfrac{1}{2}B)\ y/(\sin \tfrac{1}{2}B \cos \tfrac{1}{2}Y)];$$

therefore, if $b = \frac{1}{2}(b+y) + a$, and $\sin \frac{1}{2}Y \cos \frac{1}{2}B = \frac{1}{2}\sin \frac{1}{2}(Y+B) + \beta$,

$$\frac{\frac{1}{4}(b+y)^2 - a^2}{\frac{1}{4}\sin^2 \frac{1}{2}(Y+B) - \beta^2} = \frac{(b+y)^2}{\sin^2 \frac{1}{2}(B+Y)} = \frac{a^2}{\beta^2};$$

therefore a and β have opposite signs, *i.e.*, $(b-y)/[\sin \frac{1}{2}(B-Y)]$ is positive unless $\Delta_1 = \Delta_2$, in which case $\Delta_0 = \Delta_3$ and $\Delta_0 + \Delta_2 = \Delta_1 + \Delta_3$.

Similarly, $(c-z)/[\sin \frac{1}{2}(C-Z)]$ is positive unless $\Delta_1 = \Delta_3$; therefore

$$\frac{a-x}{\sin \frac{1}{2}(X-A)} = \frac{b-y}{\sin \frac{1}{2}(B-Y)} = \frac{c-z}{\sin \frac{1}{2}(C-Z)} = \frac{a+x}{\sin \frac{1}{2}(A+X)}$$

$$= \frac{b+y}{\sin \frac{1}{2}(B+Y)} = \frac{c+z}{\sin \frac{1}{2}(C+Z)} = \text{a positive quantity,}$$

unless $\Delta_1 = \Delta_2$ or $\Delta_1 = \Delta_3$, in which cases the signs of the second and third terms respectively must be changed.

6762. (J. Hammond, M.A.)—Prove that the sum of the series

$$\frac{\Gamma(m-n+2)}{\Gamma(m+1)\,\Gamma(m+2)} + \frac{\Gamma(m-n+3)}{\Gamma(m+2)\,\Gamma(m+3)} + \frac{\Gamma(m-n+4)}{\Gamma(m+3)\,\Gamma(m+4)} + \ldots \text{ to infinity}$$

$$= \frac{1}{\Gamma(m)\,\Gamma(n)} \int_0^1 \int_0^1 e^{xy}(1-x)^{n-1}(1-y)^{m-1} x^{m-n+1}\,dx\,dy.$$

Solution by Rev. J. Cullen and Professor Sanjána.

We clearly have

$$\frac{\Gamma(p+m-n+2)\,\Gamma(n)}{\Gamma(p+1)\,\Gamma(p+m+2)} \cdot \frac{\Gamma(p+1)\,\Gamma(m)}{\Gamma(p+m+1)}$$

$$= \int_0^1 \int_0^1 \frac{x^{p+m-n+1}(1-x)^{n-1} y^p (1-y)^{m-1}\,dx\,dy}{p\,!}$$

$$= \int_0^1 \int_0^1 \left[\frac{(xy)^p}{p\,!}\right](1-x)^{n-1}(1-y)^{m-1} x^{m-n+1}\,dx\,dy.$$

Hence, putting $p = 0, 1, 2,$ &c., we have the stated result on dividing across by $\Gamma(m)\,\Gamma(n)$.

14627. (Professor S. Sircom.)—Trace the curve

$$(x-2)^2\{(x-2)y^2 - (6-x)x^2\}^2 = 64(3-x)x^2y^2,$$

with special reference to its shape in the neighbourhood of the origin.

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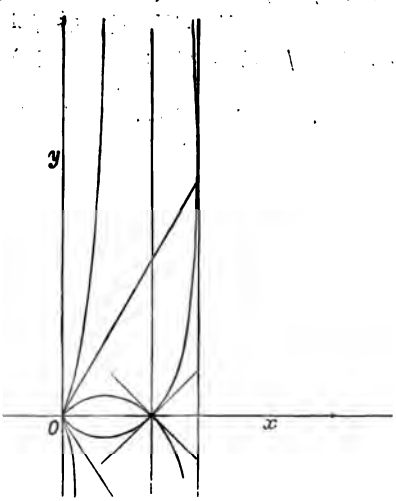
Solution by H. W. Curjel, M.A.



There are two cusps at the origin, which is a quadruple point. The tangents at the origin are

$$y^4 - 6x^2y^2 + 9x^4 = 0$$

or $(y \pm \sqrt{3}\,x)^2 = 0$.

(x^2+y^2) is a factor of the terms of highest degree. Therefore the curve is bicircular.

There is one asymptote $x-2=0$ passing through a quadruple point at infinity; (2, 0) is also a double point, the tangents at it being $x \pm y - 2 = 0$; (6, 0) is an isolated double point. There are also two imaginary double points $(4, \pm 4i)$.

These, counting as 18 (with the circular points at infinity), appear to be all the double points, as the equation may be written

$$(x-2)^2(y/x) = \pm 4\,(3-x)^{\frac{1}{2}} \pm (4-x)^{\frac{1}{2}}\,x^{\frac{1}{2}}.$$

Hence the curve appears to be of deficiency 3.

The curve is symmetrical with respect to the axis of x, and as in figure. For all real points on the curve except the acnode (6, 0), x lies between 0 and 3. $x = 3$ touches the curve at $y = \pm 3\sqrt{3}$.

14616. (R. W. D. Christie.)—Prove for primes of the form $4M-1$ the modular equation $a^2 + b = 0 \bmod p$,

where a and b are any primitive roots; *e.g.*, prime 11, roots 2, 6, 7, 8, we have $2^2+7 = 6^2+8 = 7^2+6 = 8^2+2 = 0 \bmod 11$.

I. *Solution by* H. W. Curjel, M.A.

If a is a primitive root, $p-1$ is the smallest value of n for which $a^n \equiv 1 \bmod p$ and $a^{\frac{1}{2}(p-1)} \equiv -1 \bmod p$; therefore, since $\frac{1}{2}(p-1)$ is odd, $(-a^2)^{\frac{1}{2}(p-1)} \equiv -1 \bmod p$, and hence it is clear that $p-1$ is the smallest value of n for which $(-a^2)^n \equiv 1 \bmod p$; therefore $-a^2 = b$ is also a primitive root; *i.e.*, the primitive roots may be arranged in pairs a, b, &c., so that $a^2 + b \equiv 0 \bmod p$.

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II. *Solution by the* PROPOSER.

We have, by Quest. 14668, $r^{2m-n-1}+r_1^n = 0 \bmod 4m-1$;

and, if $n = 2$, then $r^{2m-3}+r_1^2 = 0 \bmod 4m-1$, or $b+a^2 = 0 \bmod 4m-1$,

where a and b are both primitive roots, provided $2m-3$ be prime to $2(2m-1)$, which is always the case, since both 2 and $2m-1$ are prime to $2m-3$. We thus secure the connexion between the roots a and b.

E.g.—$m = 3$, $p = 11$, and

$$r^3+r_1^2 = 2^3+6^2 = 6^3+2^2 = 7^3+8^2 = 8^3+7^2 = 7+2^2 = 8+6^2 = 6+7^2 = 2+8^2 = 0 \bmod 11.$$

14713. (Prince C. DE POLIGNAC.)—If t_1, u_1 are the smallest integer solutions of the Pellian equation $t^2-Du^2 = 1$ (D a positive integer, not a square), t_n, u_n any other two solutions, show that there exists a linear substitution $x_1 = (Q_1x+S_1)/(P_1x+R_1)$ such that, if we write its nth power as $x_n = (Q_nx+S_n)/(P_nx+R_n)$, Q_n and P_n/u_1 will give the solutions t_n, u_n of the above equation.

Solution by the PROPOSER.

From LEJEUNE-DIRICHLET, *Zahlentheorie* (Braunschweig, 1863), p. 227, we have $t_n+u_n\sqrt{D} = (t_1+u_1\sqrt{D})^n$;

whence

$$t_n = t_1^n+\frac{n\,.\,n-1}{1\,.\,2}t_1^{n-2}u_1^2D+\frac{n\,.\,n-1\,.\,n-2\,.\,n-3}{1\,.\,2\,.\,3\,.\,4}t_1^{n-4}u_1^4D^2+\&c.\ldots\ldots(1),$$

$$u_n = nt_1^{n-1}u_1+\frac{n\,.\,n-1\,.\,n-2}{1\,.\,2\,.\,3}t_1^{n-3}u_1^3D + \frac{n\,.\,n-1\,.\,n-2\,.\,n-3\,.\,n-4}{1\,.\,2\,.\,3\,.\,4\,.\,5}t_1^{n-5}u_1^5D^2+\&c.\ \ldots(2).$$

On the other hand, I have shown (*Bulletin de la Société Mathématique de France*, Vol. IV., "Note sur les Substitutions Linéaires," p. 122) that, if the nth power of the linear substitution $x_1 = (bx+d)/(ax+c)$ be

$$x_n = (Q_nx+S_n)/(P_nx+R_n),$$

then for $c = b$ it reduces to $x_n = (Q_nx+S_n)/(P_nx+Q_n)$,

where

$$Q_n = b^n+\frac{n\,.\,n-1}{1\,.\,2}b^{n-2}ad+\frac{n\,.\,n-1\,.\,n-2\,.\,n-3}{1\,.\,2\,.\,3\,.\,4}b^{n-4}a^2d^2+\&c.\ldots\ldots(3),$$

$$P_n = a\left\{nb^{n-1}+\frac{n\,.\,n-1\,.\,n-2}{1\,.\,2\,.\,3}b^{n-3}ad + \frac{n\,.\,n-1\,.\,n-2\,.\,n-3\,.\,n-4}{1\,.\,2\,.\,3\,.\,4\,.\,5}b^{n-5}a^2d^2+\&c.\right\}\ldots(4).$$

Now, by hypothesis, we have in the given Question: $t_1^2-u_1^2D = 1$.

Take then the substitution

$$x_1 = (t_1x+D)/(u_1^2x+t_1) = (Q_1x+S_1)/(P_1x+Q_1),$$

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and replace in (3), (4) a, b, d respectively by u_1^2, t_1, D; we obtain

$$Q_n = t_1^n + \frac{n \cdot n-1}{1 \cdot 2} t_1^{n-2} u_1^2 D + \frac{n \cdot n-1 \cdot n-2 \cdot n-3}{1 \cdot 2 \cdot 3 \cdot 4} t_1^{n-4} u_1^4 D^2 + \&c.,$$

$$\frac{P_n}{u_1} = n t_1^{n-1} u_1 + \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} t_1^{n-3} u_1^3 D + \frac{n \cdot n-1 \cdot n-2 \cdot n-3 \cdot n-4}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} u_1^5 D^2 + \&c.$$

The right-hand members are identical with LEJEUNE-DIRICHLET formulæ (1), (2); hence $t_n = Q_n$, $u_n = P_n/u_1$.

The direct proof of the theorem would take up too much space. It results from an identical relation which can be shown to exist in connexion with the nth power of the general linear substitution $x_1 = (bx+d)/(ax+c)$, viz.,

$$aQ_n^2 + (c-b) P_n Q_n - dP_n^2 = a(bc-ad)^n,$$

where

$$x_n = (Q_n x + S_n)/(P_n x + R_n).$$

If $c = b$, this relation reduces to

$$aQ_n^2 - dP_n^2 = a(b^2-ad)^n.$$

Suppose now $b^2-ad = 1$, and take $d = D$, $b = t_1$, $a = u_1^2$, where t_1, u_1 are the smallest solutions of $t^2-u^2D = 1$; we have then

$$u_1^2 Q_n^2 - D P_n^2 = u_1^2 \quad \text{or} \quad Q_n - D(P_n/u_1)^2 = 1;$$

i.e., Q_n and P_n/u_1 are the general solutions.

If we work with the substitution $x_1 = (t_1 x + u_1 D)/(u_1 x + t_1)$, we have to replace in formula (4), a by u_1, and d by $u_1 D$. It becomes then identical with (2), and the general solutions of the Pellian equation are simply

$$t_n = Q_n, \quad u^n = P_n.$$

13989. (D. BIDDLE.)—Prove that, by joining one BROCARD point to the images of the other, we obtain the points of contact of the BROCARD ellipse with the sides of the triangle, and an easy means of construction.

Solution by W. J. GREENSTREET, M.A.

The point of contact of the BROCARD ellipse with BC is a point P, such that $\Omega'PC = \Omega PB$. Hence, if $\Omega'D$ be perpendicular to BC, and be produced to X so that X is the image of Ω' in BC, then, if ΩX intersect BC in P, we have $\angle \Omega PB = XPC = \Omega'PC$; therefore P is the point of contact required. Hence, &c.

6672. (J. GRIFFITHS, M.A.)—If four pairs of points are so arranged on a plane that any three of the pairs form a group of six points on a conic, prove that (1) the four pairs must all lie on the same conic; and hence (2) the eight points of contact of two conics with their common tangents lie on another conic.

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Solution by Professor NANSON, *and* A. M. NESBITT, M.A.

Denoting the joins of the four pairs by $\alpha, \beta, \gamma, \delta$, let S be the conic on which lie the pairs α, β, γ; then the conics on which lie the pairs β, γ, δ; γ, α, δ; α, β, δ must be $S + \lambda\beta\gamma$, $S + \lambda\gamma\alpha$, $S + \lambda\alpha\beta$. If now $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$ denote the points on δ, we have

$$S_1 + \lambda\beta_1\gamma_1 = 0, \quad S_1 + \mu\gamma_1\alpha_1 = 0, \quad S_1 + \nu\alpha_1\beta_1 = 0,$$
$$S_2 + \lambda\beta_2\gamma_2 = 0, \quad S_2 + \mu\gamma_2\alpha_2 = 0, \quad S_2 + \nu\alpha_2\beta_2 = 0,$$

where S_1, S_2 are the values of S at the points on δ. From these equations it follows that $\lambda = 0$, $\mu = 0$, $\nu = 0$; so that the four pairs lie on the conic S.

If now the eight points are the contacts of the common tangents to two conics, it follows, from CARNOT'S theorem, that the contacts of any three tangents are on a conic, and therefore the contacts of all four are on a conic.

12881. (Professor E. LEMOINE.)—Soient H l'orthocentre d'un triangle ABC et O le centre du cercle circonscrit; les hauteurs AH, BH, CH rencontrent la circonférence ABC en A_1, B_1, C_1; la droite HO rencontre BC, CA, AB en A′, B′, C′; enfin A_2, B_2, C_2 désignent les symétriques de A, B, C par rapport à O. Démontrer que (1) les droites A_2A', B_2B', C_2C' concourent en un point M de la circonférence ABC; (2) les points M, H et le point de TARRY sont en ligne droite; (3) les droites A_1A', B_1B', C_1C' se coupent au point de STEINER.

Solution by Professor A. DROZ-FARNY.

(*Mathesis*, 1897, p. 175). The symmetrics, with respect to the sides of the triangle, of a transversal through H intersect at a point on the circumcircle, the SIMSON line for which is parallel to the transversal. The lines $A'A_1$, ... intersect on the circumcircle at R, and the SIMSON line of R is parallel to OH. Hence R is the STEINER point of ABC.

By the converse of a theorem of M. DÉPREZ (*Mathesis*, v., p. 230), the lines A_2A', ... intersect at M on the circumcircle. Let HM cut the circle in T. Then in the inscribed hexagon AA_1RTMA_2, RT must concur at O with AA_2 and HA′, and, R, T being at the ends of a diameter, T is the TARRY point of ABC.

14374. (H. J. WOODALL, A.R.C.S.)—Solve $x^{23} \equiv 1 \pmod{1933}$.

Solution by Lt.-Col. ALLAN CUNNINGHAM, R.E.

This is easy to solve, if (1) any primitive root of $p = 1933$ is known, or (2) if any root (a) is known such that $a^\xi \equiv 1 \pmod{1933}$, where ξ contains the factor 23.

Now REUSCHLE'S *Neue Zahlentheoretische Tabellen*, Stuttgart, 1856, gives $2^{644} \equiv +1 \pmod{1933}$, whence

$$2^{322} \text{ or } (2^{14})^{23} \equiv -1, \text{ or } 16384^{23} \equiv 920^{23} \equiv -1, \text{ and } 1013^{23} \equiv +1;$$

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so that $x = 1013$ is one root. The other roots (22 in all) are given as the least residues of x^r, where $r = 1, 2, 3$, &c., ... 22. In this way the roots are found to be $-920, -254, -213, +727, -22, +910, -211, +820, -530, +484, -690, +776$, &c.

[The PROPOSER adds the following:—"Otherwise (3). Take a small number (a), and raise it to the power η, where $23\eta = 1932$, giving $\eta = 84$; then the residue of a^η (mod 1933) is a root of the given equation. In this case 7 is convenient. Now $7^{84} \equiv 1722$ for modulus = 1933; consequently 1722 is a root. The 22 roots arranged consecutively as powers of the smallest are 62, 1911, 569, 484, 1013, 950, 910, 363, 1243, 1679, 1649, 1722, 449, 776, 1720, 325, 820, 582, 1290, 727, 615, 1403. It is interesting to notice that, because 484 and 363 are both roots, so also will be the ratio 4 : 3. In fact,

$$4^{23} - 3^{23} = 70{,}274{,}600{,}998{,}837 = 47.1933.17389.44483."]$$

14668. (ROBERT W. D. CHRISTIE.)—Prove the following modular equations for any prime:—

$$r^{2m-n} + r_1{}^n \equiv 0 \bmod p = 0 \bmod 4m+1,$$

$$r^{2m-n-1} + r_1{}^n = 0 \bmod p = 0 \bmod 4m-1,$$

r and r_1 being associated primitive roots, and n any integer whatever.

I. *Solution by* H. W. CURJEL, M.A.

Since $r^{\frac{1}{2}(p-1)} \equiv -1 \equiv -(rr_1)^n \bmod p$, $r^{\frac{1}{2}(p-1)-n} + r_1{}^n \equiv 0 \bmod p$;

i.e., $r^{2m-n} + r_1{}^n \equiv 0 \bmod 4m+1$ and $r^{2m-n-1} + r_1{}^n \equiv 0 \bmod 4m-1$.

II. *Solution by* Lt.-Col. ALLAN CUNNINGHAM, R.E.

(1) When $p = 4m+1$, then $r^{2m} \equiv r_1{}^{2m} \equiv -1 \pmod p$; also $r.r_1 \equiv +1$; therefore $r_n.r_1{}^n \equiv +1 \equiv -r^{2m} \pmod p$; therefore $r_1{}^n \equiv -r^{2m-n}$, and $r^{2m-n} + r_1{}^n \equiv 0 \pmod p$.

(2) When

$$p = 4m-1 = 2(2m-1)+1, \text{ then } r^{2m-1} \equiv r_1{}^{2m-1} \equiv -1 \pmod p;$$

also $rr_1 \equiv +1$; therefore $r^n.r_1{}^n \equiv +1 \equiv -r^{2m-1} \pmod p$; therefore $r_1{}^n \equiv -r^{2m-n-1}$, and $r^{2m-n-1} + r_1{}^n \equiv 0 \pmod p$.

6730. (J. R. HARRIS, M.A.)—Prove that (1) the pedal of a helix with regard to any point on its axis is a curve lying on a hyperboloid of revolution; and (2), if the pitch of the helix be $\frac{1}{2}\pi$, this curve cuts all the generators of one system of the hyperboloid at right angles.

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Solution by Rev. J. CULLEN.

(1) Taking the origin for the given point, we have for the helix $x' = a\cos\theta$, $y' = a\sin\theta$, $z' = k\theta$, where $k = a\tan\alpha$, α being the pitch. Now $\{(a-lt), (b-mt), (c-nt)\}$, where $t = \Sigma al/\Sigma l^2$, is the foot of the perperpendicular from the origin on the line $(x-a)/l = (y-b)/m = (z-c)/n$; for the tangent we have $a=x'$, &c., $l=dx'/d\theta$, &c.; whence, substituting, we find the coordinates of the required curve as follows:—

$$ax = a^2\cos\theta + zk\sin\theta, \quad ay = a^2\sin\theta - zk\cos\theta, \quad z = a^2k\theta/(a^2+k^2);$$

so that on the elimination of θ we get $(x^2+y^2)/a^2 - k^2z^2/a^4 = 1$, a hyperboloid of revolution.

(2) When $k = a$, $\alpha = \frac{1}{4}\pi$, and the direction cosines of the generator $x-z = \mu(a+y)$, $x+z = 1/\mu(a-y)$ may be taken as L, 1, $-$M, where

$$L = \tfrac{1}{2}(\mu - 1/\mu) = -(az+xy)/(a^2-y^2), \quad M = \tfrac{1}{2}(\mu + 1/\mu) = (ax+yz)/(a^2-y^2);$$

also the direction cosines of the tangent are $dx/d\theta$, &c. Hence we find on substitution that $L\,dx/d\theta + dy/d\theta - M\,dz/d\theta = 0$, or the curve cuts this system of generators at right angles.

14648. (Prebendary WHITWORTH, M.A.)—If a straight line be divided at random into four parts, the expectation of the sum of the squares on the greatest and least parts is to the expectation of the sum of the squares on the other two parts as 53 to 19. (Solution by Algebra.)

Solution by Rev. T. ROACH, M.A.

If $a\alpha + b\beta + c\gamma + \ldots = S$, where α, β, γ are variables, then, n being the number of variables, $\in$ the expectation of any term in the expansion of $(a\alpha + b\beta + c\gamma + \ldots)^m$,

$$\in \alpha^p\beta^q\gamma^r \ldots = \frac{S^{p+q+r} \ldots (n-1)!\,p!\,q!}{a^p b^q c^r \ldots (n-1+p+q-\ldots)!},$$

where $p+q+\ldots = m$ [*Choice and Chance*].

Now, let a stick be divided at random into four parts α, $\alpha+\beta$, $\alpha+\beta+\gamma$, and $\alpha+\beta+\gamma+\delta$. Find the ratio of

$$\in\alpha^2 + \in(\alpha+\beta+\gamma+\delta)^2 : \in(\alpha+\beta)^2 + \in(\alpha+\beta+\gamma)^2.$$

Here $a = 4$, $b = 3$, $c = 2$, $d = 1$, $n = 4$, $m = 2$,

and $$\in\alpha^p\beta^q = \frac{S^2\,3!}{5!}\,\frac{p!\,q!}{a^p b^q}.$$

Now the variable factor for the term involving

$$\alpha^2 = \frac{2}{4^2}, \quad \beta^2 = \frac{2}{3^2}, \quad \gamma^2 = \frac{2}{2^2}, \quad \delta^2 = \frac{2}{1};$$

for $\alpha\beta = \frac{1}{4.3}$, $\alpha\gamma = \frac{1}{4.2}$, $\alpha\delta = \frac{1}{4.1}$, $\beta\gamma = \frac{1}{3.2}$, $\beta\delta = \frac{1}{3.1}$, $\gamma\delta = \frac{1}{2.1}$;

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therefore $\in a^2 + \in (\alpha+\beta+\gamma+\delta)^2 : \in (\alpha+\beta)^2 + \in (\alpha+\beta+\gamma)^2$

$$= \frac{1}{4^2}+\frac{1}{4^2}+\frac{1}{3^2}+\frac{1}{2^2}+1+\frac{1}{4.3}+\frac{1}{4.2}+\frac{1}{4.1}+\frac{1}{3.2}+\frac{1}{3.1}+\frac{1}{2.1}$$
$$: \frac{1}{4^2}+\frac{1}{3^2}+\frac{1}{4.3}+\frac{1}{4^2}+\frac{1}{3^2}+\frac{1}{2^2}+\frac{1}{4.3}+\frac{1}{4.2}+\frac{1}{3.2}$$

$$= 9+9+16+36+144+12+18+36+24+48+72$$
$$: 9+16+12+9+16+36+12+18+24$$
$$= 424 : 152 = 53 : 19.$$

14408. (E. W. Rees, B.A.)—If the feet of the perpendiculars from K, G, the symmedian point and centre of gravity of the triangle ABC, on the sides BC, CA, AB, respectively, be XX′, YY′, ZZ′, prove that the radius of the circle circumscribing the hexagon XX′YY′ZZ′ is $\frac{4}{3}\{(m_1m_2m_3)/(a^2+b^2+c^2)\}$, where m_1, m_2, m_3 are the medians of the triangle ABC.

Solution by the Proposer.

K and G are isogonal points; *i.e.*, if $\angle KAB = \theta_1$ and $\angle KAC = \theta_2$, then $\angle GAB = \theta_2$ and $\angle GAC = \theta_1$.

Let R be the radius of the circle circumscribing the triangle XYZ. Therefore, from Fig. 1, $R = \frac{1}{2}(XY/\sin XZY)$; and, from quadrilateral KXCY, $XY = CK \sin C$.

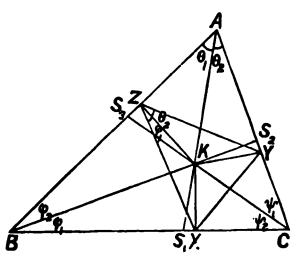

Fig. 1.

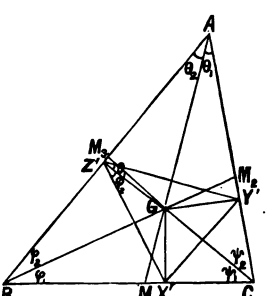

Fig. 2.

Now $\sin XZY = \sin(\phi_1+\theta_2) = \sin AGB$ (from Fig. 2).

But $\Delta AGB = \frac{1}{3}\Delta = \frac{1}{2}(\frac{2}{3}m_1 \frac{2}{3}m_2 \sin AGB)$,

and CK can be easily shown to be equal to $2abm_3/(a^2+b^2+c^2)$. Therefore

$$R = \frac{2abm_3 \sin C}{2(a^2+b^2+c^2)} \Big/ \frac{3\Delta}{2m_1m_2} = \frac{2\Delta m_3}{a^2+b^2+c^2}\,\frac{2m_1m_2}{3\Delta} = \frac{4m_1m_2m_3}{3(a^2+b^2+c^2)}.$$

The same value is obtained for R as considered circumscribing the triangle X′Y′Z′ of Fig. 2.

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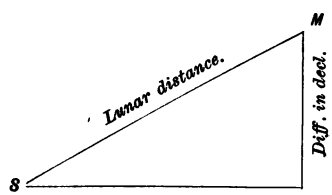
6707. (Sir W. Thomson, F.R.S., now Lord Kelvin.)—On a clear night you are suddenly placed in a boat somewhere hundreds of miles from land in the northern hemisphere, on either the Atlantic or the Pacific, with a *Nautical Almanac* (and a light to read it), knowing the month, but not the day or hour: you see the Moon with a bright star close beside her, as near the meridian as you can judge by reference to the Pole-star. Find the correct date, and tell whether you are in the Atlantic or the Pacific.

Solution by the Proposer.

By aid of a string and foot-rule, or a tape measuring line, measure correctly within one degree the angular distance of the star from the Moon. Judge the age of the Moon from her appearance, and look in the *Nautical Almanac* among the four or five possible days, for a lunar distance approximately equal to that which you have measured, helping yourself by noting whether more than one star was at all near the Moon at the time of your observation. This gives you your Greenwich date, that is to say, the Greenwich day and hour of the time of your observation correctly within about three hours. Find in the *Nautical Almanac* the Moon's right ascension at that time. As the Moon was nearly on the meridian at the time, you have your longitude correctly enough to tell whether you are in the Atlantic or the Pacific.

This was a rider on a question in which the principle of the method of lunars for determining longitude at sea was fully discussed, and also a method of measuring by a person in a boat, without other instruments than a measuring tape, or a piece of string and a foot-rule, the distance between two celestial objects not more than 30° apart correctly within one degree.

[Mr. J. Macleod, M.A., had previously sent the following:—The night being clear, our boatman may be supposed to recognize the star in proximity to the Moon, and on looking at his *Nautical Almanac* he will at once discover the day and hour at which the star is thus near the Moon. Knowing Greenwich time, he can tell where the Sun is relatively to the Moon. Knowing also the declinations of Sun and Moon, he can calculate the distance of the Sun from his meridian (on which the Moon is supposed to be) and hence his longitude. In the circumstance, logarithms probably being disallowed, a mental calculation could be made by treating the actual spherical triangle as plane.]



14624 & 14422. (14624.) (Professor E. Lemoine.)—Soient en coordonnées normales $a\,(m-n)$, $b\,(n-l)$, $c\,(l-m)$ les coordonnées d'un point M de la droite de Lemoine; soit $A_1B_1C_1$ le triangle formé par les pieds des perpendiculaires abaissées de M respectivement sur BC, CA, AB; soit un triangle A′B′C′ homothétique à $A_1B_1C_1$. (1) Si AA′, BB′, CC′ concourent

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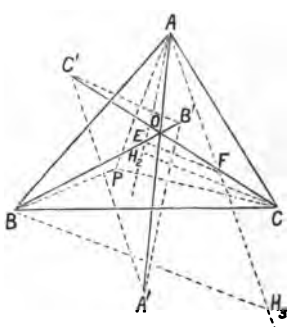
en O, le lieu de O est une hyperbole équilatère circonscrite à ABC ; si AB′, BC′, CA′ concourent en O′, ou AC′, BA′, CB′ concourent en O″, les lieux de O′ et de O″ sont aussi des hyperboles équilatères circonscrites à ABC. (2) Donner les équations des trois courbes.

(14422.) (Professor E. LEMOINE.)—Soient deux triangles ABC, A′B′C′ dont j'appelle les côtés a, b, c ; a', b', c'. Je suppose ABC fixe. Ils sont tels que a et a' sont perpendiculaires ainsi, que b et b' et que c et c'. Cela posé, si ABC et A′B′C′ sont homologiques, le lieu du centre O d'homologie est une conique D circonscrite à ABC ; si ABC et B′C′A′ sont homologiques, le lieu est une conique D′ circonscrite à ABC ; si ABC et C′A′B′ sont homologiques, le lieu est une conique D″ circonscrite à ABC. Les trois coniques D, D′, D″ ont pour point commun le point de TARRY du triangle ABC. Déterminer le triangle A′B′C′ qui fait en même temps des centres d'homologie O, O′, O″ avec ABC, c'est à dire que ABC et A′B′C′ soient trihomologiques. Remarquons que par leur construction ABC et A′B′C′ sont toujours triorthologiques.

Solution by C. E. YOUNGMAN, M.A.

In general, if any triangle A′B′C′, having B′C′, C′A′, A′B′ perpendicular to AP, BP, CP, respectively, be also such that AA′, BB′, CC′ meet at O, the conic ABCPO is a rectangular hyperbola. For, if we take H_2, H_3, the orthocentres of PCA and PAB, and draw AH_2E, AH_3F to meet BO, CO at E, F, we have



$$OE : OB' = OA : OA' = OF : OC',$$

and EF is parallel to B′C′, or CH_2, or BH_3; therefore, by the converse of PASCAL's theorem, the points B, O, C, H_2, A, H_3 lie on a conic, which, having five points common with the rectangular hyperbola $ABCPH_2H_3$, coincides with it.

Now take for P the point isogonal to M in ABC; its coordinates are given by $a\alpha(m-n) = b\beta(n-l) = c\gamma(l-m)$. Then AP, BP, CP are perpendicular to B_1C_1, C_1A_1, A_1B_1, respectively, and therefore to B′C′, C′A′, A′B′. Hence the locus of O is the rectangular hyperbola ABCP, whose equation is

$$\begin{vmatrix} \alpha^{-1} & \beta^{-1} & \gamma^{-1} \\ \cos A & \cos B & \cos C \\ a(m-n) & b(n-l) & c(l-m) \end{vmatrix} = 0,$$

or

$$\beta\gamma(la - mc\cos B - nb\cos C) + \ldots + \ldots = 0 \quad \ldots\ldots\ldots\ldots \quad (1).$$

Again, it is easily verified that the point P_1, for which $a\alpha(l-m) = b\beta(m-n) = c\gamma(n-l)$, is such that AP_1, BP_1, CP_1 are parallel to BP, CP, AP, and therefore perpendicular to C′A′, A′B′, B′C′; accordingly O′ lies on the rectangular hyperbola $ABCP_1$, whose equation will be had by writing n, l, m for l, m, n in (1). Similarly, with m, n, l instead of l, m, n, we should get a point P_2 and the locus of O″.

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Quest. 14422 is a particular case ($m = n$); M is on BC, and P, P_1, P_2 are at A, B, C. (The "point de TARRY" part must be a slip.)

It is plain from a figure that, if we make B'C', C'A', A'B' pass through A, C, B, respectively (or C, B, A; or B, A, C), O, O', O'' will coincide with A', C', B' (or B', A', C'; or C', B', A'). Writing the equations of the loci of O, O', O'' in the forms $S_1 \equiv a_1\beta\gamma + b_1\gamma\alpha + c_1\alpha\beta = 0$, and so on, I find for the locus of A' (when ABC and A'B'C' are trihomologic) the equation

$$\alpha\beta\gamma a_1 b_3 c_2 + (\alpha b_2 + \beta a_2)(\beta c_1 + \gamma b_1)(\gamma a_3 + \alpha c_3) = 0,$$

or

$$\alpha\beta\gamma a_1 b_3 c_2 + \frac{\partial S_1}{\partial \alpha}\frac{\partial S_3}{\partial \beta}\frac{\partial S_2}{\partial \gamma} = 0.$$

14705. (Professor NEUBERG.) — Etant donné un triangle ABC, appelons a', b', c' les distances B'C', C'A', A'B' des projections A', B', C' d'un point M sur les côtés BC, CA, AB. Trouver le lieu des points M lorsque a', b', c' vérifient la relation donnée $ka'^2 + lb'^2 + nc'^2 = p$. Trouver également le lieu décrit par le centre de gravité du triangle A'B'C'.

Solution by CONSTANCE I. MARKS, B.A.

With usual notation, the sides of given triangle and its area (known quantities by hypothesis) are a, b, c, Δ.

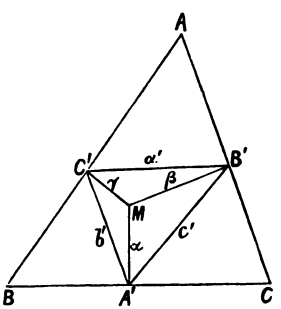
Take ABC as triangle of reference, and let α, β, γ be the trilinear coordinates of any position of M. Then

$$a'^2 = \beta^2 + \gamma^2 + 2\beta\gamma\cos A \text{.....(i.)},$$
$$b'^2 = \gamma^2 + \alpha^2 + 2\gamma\alpha\cos B \text{ ... (ii.)},$$
$$c'^2 = \alpha^2 + \beta^2 + 2\alpha\beta\cos C \text{ ... (iii.)}.$$

Multiply (i.) by k, (ii.) by l, (iii.) by n, and add. Remembering given relation

$$p = ka'^2 + lb'^2 + nc'^2,$$

and that $\dfrac{a\alpha + b\beta + c\gamma}{2\Delta} = 1$,



the result is

$$\frac{p(a\alpha + b\beta + c\gamma)^2}{4\Delta^2} = (l+n)\alpha^2 + (n+k)\beta^2 + (k+l)\gamma^2 + 2k\cos A\beta\gamma + 2l\cos B\gamma\alpha + 2n\cos C\alpha\beta \text{...(iv.)};$$

therefore the locus of M is a conic.

[The PROPOSER solves the second part of the Question as follows:— "Soient M' le centre de gravité de A'B'C', et α', β', γ' ses coordonnées; les distances de B' et C' au côté BC sont égales à $\alpha + \beta\cos C$, $\alpha + \gamma\cos B$; par conséquent $3\alpha' = 2\alpha + \beta\cos C + \gamma\cos B$.

Par analogie, $3\beta' = \alpha\cos C + 2\beta + \gamma\cos A$, $3\gamma' = \alpha\cos B + \beta\cos A + 2\gamma$. De ces trois égalités on peut tirer α, β, γ en fonction de α', β', γ'; en substituant les valeurs dans l'équation (iv.), on aura l'équation du lieu de M', qui est une conique."]

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6724. (E. B. ELLIOTT, M.A.)—If $a_1, a_2, \ldots, a_{2r}$ be the $2r$ imaginary $(2r+1)$th root of unity, and if

$$c_n = \frac{a_1^{n+2r-1}}{(a_1-a_2)(a_1-a_3)\ldots(a_1-a_{2r})} + \frac{a_2^{n+2r-1}}{(a_2-a_1)(a_2-a_3)\ldots(a_2-a_{2r})} + \ldots,$$

prove that $c_n = 2\left\{c_0c_{2n} - c_1c_{2n-1} + c_2c_{2n-2} - \ldots + (-1)^n c_n^2\right\}$,

n being any positive integer.

Solution by G. D. WILSON, B.A.

Since $1, a_1, a_2, \ldots, a_{2r}$ are roots of $x^{2r+1}-1=0$,

therefore $a_s-1, a_s-a_1, \ldots, a_s-a_{2r}$ are roots of $(x-a_s)^{2r+1}+1=0$;

and therefore $(a_s-1)(a_s-a_1)\ldots(a_s-a_{2r}) = (2r+1)\,a_s^{2r}$,

the factor (a_s-a_s) being, of course, omitted on the left-hand side

$$\therefore \quad c_n = \sum_{s=1}^{2r} \frac{a_s^{n+2r-1}}{(2r+1)\,a_s^{2r}}(a_s-1) = \frac{1}{2r+1}\sum_{s=0}^{2r}(a_s^n - a_s^{n-1}), \text{ where } a_0 = 1.$$

Again, $(1-a_sx)$ is a root of $(1-y)^{2r+1}-x^{2r+1}=0$; so

$$\sum_{s=0}^{2r}\frac{1}{1-a_sx} = \frac{2r+1}{1-x^{2r+1}};$$

therefore $$\frac{1-x}{1-x^{2r+1}} = \frac{1}{2r+1}\sum_{s=0}^{2r}\frac{1-x}{1-a_sx} = \frac{1}{2r+1}\sum_{n=0}^{\infty}x^n\sum_{s=0}^{2r}(a_s^n - a_s^{n-1}),$$

since $$\sum_{0}^{2r}\frac{1}{a_s} = 0;$$

and therefore $$\frac{1-x}{1-x^{2r+1}} = \sum_0^\infty c_nx^n \quad \text{and} \quad \frac{1+x}{1+x^{2r+1}} = \sum_0^\infty (-1)^n c_nx^n;$$

therefore $$2\left\{c_0c_{2n} - c_1c_{2n-1} + \ldots + (-1)^n e\right\} - (-1)^n c_n^2$$

$$= \text{coefficient of } x^{2n} \text{ in } \frac{1-x^2}{1-x^{4r+2}} = c_n.$$

[This result differs slightly from that given in the Question.]

14649. (H. W. CURJEL, M.A.)—If in hyperspace of n dimensions we call the hypersphere of highest possible dimensions simply a sphere, and a space of $(n-1)$ dimensions simply a plane, and the figure corresponding to a triangle in space of 2 dimensions and a tetrahedron in space of 3 dimensions a $(n+1)$-hedroid, prove that the following theorem is true if n is even and not true if n is odd:—In space of n dimensions the $n+2$ spheres circumscribing the $n+2$ $(n+1)$-hedroids determined by $n+2$ planes meet in a point.

Solution by the PROPOSER.

Let ABCD, &c., be the $(n+1)$-hedroid determined by $n+1$ of the planes, and take it as the $(n+1)$-hedroid of reference, using barycentric coordinates $(\alpha, \beta, \gamma, \delta, \text{&c.})$.

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The equation to the sphere circumscribing the $(n+1)$-hedroid of reference is $$S \equiv \Sigma AB^2 a\beta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Let $l\alpha + m\beta + n\gamma + p\delta + \ldots = 0$ be the equation to the remaining plane. Then the equation to the sphere circumscribing the $(n+1)$-hedroid formed by all the planes except $\alpha = 0$ is easily found to be

$$S + (\alpha + \beta + \gamma + \delta + \ldots)\left(\frac{AB^2 m\beta}{l-m} + \frac{AC^2 n\gamma}{l-n} + \frac{AD^2 p\delta}{l-p} + \ldots\right) = 0 \ldots\ldots (2);$$

therefore the equation to the radical plane of (1) and (2) is

$$\frac{AB^2 m\beta}{l-m} + \frac{AC^2 n\gamma}{l-n} + \frac{AD^2 p\delta}{l-p} + \ldots = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (a).$$

Similarly the equations to the radical planes of S and the other spheres are

$$\frac{AB^2 l\alpha}{m-l} + \frac{BC^2 n\gamma}{m-n} + \frac{BD^2 p\delta}{m-p} + \ldots = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (b),$$

&c.

The eliminant of the equations (a), (b), &c., is clearly a skew-symmetric determinant of degree $n+1$ with the columns multiplied by l, m, n, p, &c., and therefore vanishes if n is even, but not if n is odd, except for special values of l, m, n, p, &c., AB, BC, &c., for AB, BC, &c., are connected by no equations, only by inequalities. Hence in general (*i.e.*, excluding such particular cases) the radical planes meet or do not meet in a point according as n is even or odd. Hence, if n is odd, the theorem is not true.

If n is even, by multiplying (a) by α, (b) by β, &c., and adding, we get $-S = 0$. Hence the common point of the radical planes lies on S; therefore the $n+2$ spheres meet in a point, *i.e.*, the theorem is true if n is even. If we invert the figure with respect to a point in the space, the theorem is unaltered except that the $n+2$ planes are replaced by $n+2$ spheres passing through a point, and we get an entirely symmetrical figure consisting of $2n+4$ spheres passing $n+2$ at a time through 2^{n+1} points, 2^n lying on each sphere. Inversion with respect to a point outside the space leads to a similar theorem on a hypersphere.

14633. (D. Biddle.)—A certain group of cubic equations of the form $x^3 + qx + r = 0$, where q and r are both minus quantities, is capable of transformation into $y^3 - 2y - 2 = 0$, by taking $y = mx$. Prove that, with the same values of q and r respectively, $z^3 + z^2 + \frac{1}{2}(q \pm r) = 0$ and $z^3 - z^2 - \frac{1}{2}(q \mp r) = 0$ have a root in common. Also, from the foregoing law, deduce a method of finding either of the two equations in z, having given the fellow to it only.

Solution by Lt.-Col. Allan Cunningham, R.E.

$y = mx$ converts $x^3 + qx + r = 0$ into $y^3 - 2y - 2 = 0$, if $m^2 q = -2$, $m^3 r = -2$; or $q = -2\mu^2$, $r = -2\mu^3$, where $\mu = 1 + m$.

$$z^3 + z^2 + \tfrac{1}{2}(q \pm r) = 0 \quad \text{becomes} \quad (z^3 \mp \mu^3) + (z^2 - \mu^2) = 0,$$
$$z^3 - z^2 - \tfrac{1}{2}(q \mp r) = 0 \quad \text{becomes} \quad (z^3 \mp \mu^3) - (z^2 - \mu^2) = 0,$$

and these have the common root $z = \pm\mu$.

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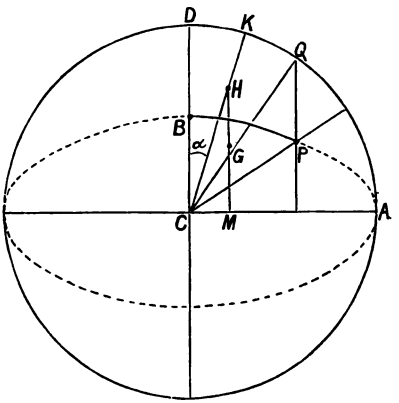
(2) If the first equation in z be given, this fixes $(q \pm r) = 2a$, suppose; hence $\mu^2 \pm \mu^3 = -a$. Solving this for μ gives μ, m, q, r, and thus fixes $(q \mp r)$, thus giving the second equation also. Similarly, from the second equation (supposed given) the first is deducible.

14199. (Professor Umes Chandra Ghosh.)—ACB is a quadrant of an ellipse, and CP any central radius vector. If x and y are the distances of the centre of gravity of the segment BCP of the ellipse from the major and minor axes, show that

$$ay = bx \cot\left\{\frac{2ab^2x}{3(a^2y^2 + b^2x^2)}\right\}.$$

Solution by F. L. Ward, B.A.; *and* Professor N. Bhattacharyya.



Let DQA be the corresponding quadrant of the major auxiliary circle, DCQ corresponding to BCP.

Draw CK bisecting $\angle$ DCQ. Take

$$CH = \tfrac{2}{3}\frac{a \sin \alpha}{\alpha},$$

where $\angle DCK = \alpha$.

Draw HM perpendicular to CA, and take G so that

$$\frac{GM}{HM} = \frac{b}{a}.$$

Then G is the point (x, y), and

$$x = \tfrac{2}{3}\frac{a \sin^2 \alpha}{\alpha},$$

$$y = \frac{b}{a}HM = \tfrac{2}{3}\frac{b \sin \alpha \cos \alpha}{\alpha}; \quad \text{therefore } x = \frac{a(1 - \cos DCQ)}{3\alpha},$$

and $$y = \frac{b \sin DCQ}{3\alpha}; \quad \text{therefore } \frac{ay}{bx} = \cot \alpha,$$

and $$1 - \frac{3ax}{a} = \cos DCQ, \quad \frac{3ay}{b} = \sin DCQ;$$

therefore $$\left(1 - \frac{3ax}{a}\right)^2 + \left(\frac{3ay}{b}\right)^2 = 1; \text{ therefore } -\frac{6ax}{a} + 9a^2\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = 0;$$

therefore $$\alpha = \frac{2ab^2x}{3(a^2y^2 + b^2x^2)}; \text{ therefore } ay = bx \cot\left\{\frac{2ab^2x}{3(a^2y^2 + b^2x^2)}\right\}.$$

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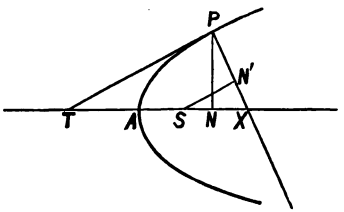
13280. (Professor BARISIEN.)—Le lieu des projections du foyer d'une parabole sur les normales est une parabole (théorème connu). Démontrer que le lieu des projections du foyer d'une parabole sur les normales à sa développée est une cubique.

Solution by W. J. GREENSTREET, M.A.



Let the tangent and normal at P meet the axis at T and X, and let PN, SN′ be perpendiculars to TX and PX respectively. Then, since S is mid-point of TX, N′ bisects PX, and the ordinate of P is double that of N′. Also NX = 2AS; therefore the abscissa of N′ is ½NX, *i.e.*, AS. Hence the locus of N′ is a parabola, vertex S, and its equation is $y^2 = ax$, a known result. If we take as origin the image of A in S, the evolute is $54ay^2 = 8x^3$, and any point on the curve is $(\frac{27}{4}at^2, \frac{27}{4}at^3)$. Normal at this point is $24my + 16x - 54at^2(3t^2+4) = 0$. The perpendicular from $(-a, 0)$ is $2y - 3mn + \frac{3}{2}at = 0$. Eliminating t, and referring to S as origin, we have $ay^2 = x^2(x-a)$.

14603. (J. A. THIRD, M.A., D.Sc.)—U is a straight line in the plane of a triangle ABC, Δ_1 is the conic with respect to which U is the polar of the centroid, and ABC self-polar. Prove the following statements:—(i.) If a variable in-conic, touching BC, CA, AB at X, Y, Z, has its centre on U, AX, BY, CZ cointersect on a circum-conic Δ_2. (ii.) If a variable circum-conic has its centre on U, and if LMN be the triangle formed by the tangents to it at A, B, C, AL, BM, CN cointersect on a conic Δ_3, passing through the mid-points of the sides of ABC. (iii.) Δ_1, Δ_2, Δ_3 are similar and similarly situated, and intersect on U. (iv.) The centre of Δ_3 bisects the distance between the centres of Δ_1 and Δ_2; the line of centres passes through the centroid; the centroid and the centre of Δ_1 are the centres of similitude of Δ_2 and Δ_3. (v.) Δ_1, Δ_2, Δ_3 are parabolas, hyperbolas, or ellipses, according as U touches, cuts, or does not cut in real points the maximum inscribed ellipse of ABC; they are circles when U is the line $\Sigma a \cos A = 0$, and equilateral hyperbolas when U passes through the symmedian point; generally, their eccentricity is constant, as U envelops a conic having double contact with the maximum inscribed ellipse on the line $\Sigma a \cos A = 0$. (vi.) If a variable circum-conic pass through the centre of Δ_1, its centre-locus is Δ_3, and if L′M′N′ be the triangle formed by the tangents to it at A, B, C, AL′, BM′, CN′ cointersect on U.

Solution by ERNEST M. ANDERSON, M.A., B.Sc.

(i.) Using areal coordinates with ABC as triangle of reference, the coordinates of the centroid are ⅓, ⅓, ⅓. Any conic for which ABC is self-polar can take the form $ax^2 + by^2 + cz^2 = 0$ .....(1).

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The line U will be $ax+by+cz=0$ .....(2).

The equation to an in-conic is of the form

$$p^2x^2+q^2y^2+r^2z^2-2qryz-2rpzx-2pqxy=0 \text{.....(3).}$$

AX, BY, CZ are given by making $x=0$, &c. in above equation, which gives $qy-rz=0, \quad rz-px=0, \quad px-qy=0.$

They intersect in the point where $px=qy=rz=\epsilon$ .....(4).

The centre of the in-conic (3) is easily found to be given by

$$x:y:z=q+r:r+p:p+q.$$

If this lie on U, $a(q+r)+b(r+p)+c(p+q)=0,$

which with (4) gives

$$a(1/y+1/z)+b(1/z+1/x)+c(1/x+1/y)=0,$$

$$(b+c)yz+(c+a)zx+(a+b)xy=0 \text{.....(5),}$$

the equation to a circum-conic Δ_2.

(ii.) The equation to any circum-conic is

$$lyz+mzx+nxy=0 \text{.....(6),}$$

whose centre is given by

$$x:y:z=l(m+n-l):m(n+l-m):n(l+m-n).$$

The condition that this lie on U is

$$\Sigma(b+c)mn-\Sigma al^2=0 \text{ ..... (7).}$$

The tangents to (6) at A, B, and C are

$$ny+mz=0, \quad lz+nx=0, \quad mx+ly=0,$$

from which AL, BM, CN are found by elimination to be

$$mz-ny=0, \quad nx-lz=0, \quad ly-mx=0.$$

They intersect where $x:y:z=l:m:n.$

This, with (7), gives $\Sigma(b+c)yz-\Sigma ax^2=0$ .....(8),

the equation of a conic (Δ_3) which passes through the points $0, \frac{1}{2}, \frac{1}{2}$; $\frac{1}{2}, 0, \frac{1}{2}$; $\frac{1}{2}, \frac{1}{2}, 0$, the mid-points of the side of the triangle.

(iii.) If we substitute, in (1), (5), and (8), $z=1-x-y$, they become

$$(a+c)x^2+(b+c)y^2+2cyz+\text{terms of lower degree}=0,$$

$$-(a+c)x^2-(b+c)y^2-2cyz+\text{&c.}=0,$$

and $$-2(a+c)x^2-2(b+c)y^2-4cyz+\text{&c.}=0.$$

If we now transform to Cartesian coordinates, by means of a linear substitution, the coefficients of terms of the second degree will remain proportional in all three cases.

Hence Δ_1, Δ_2, and Δ_3 are similar and similarly situated. Where Δ_1 and Δ_2 intersect, we get, from (1) and (5),

$$\Sigma ax^2+\Sigma(b+c)yz=0, \quad (ax+by+cz)(x+y+z)=0.$$

Thus these conics intersect on U, and it is obvious from the equations that any points of intersection are common to all three conics.

(iv.) Let (x_1), (x_2), (x_3) be the coordinates of centres of Δ_1, Δ_2, Δ_3. Then (x_1) is given by

$$ax_1=by_1=cz_1, \quad x_1=bc/(bc+ca+ab), \quad y_1=\text{&c.} \text{.....(9);}$$

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(x_2) is given by

$$\left.\begin{aligned}(c+a)\,z_2+(a+b)\,y_2 = (a+b)\,x_2+(b+c)\,z_2 = (b+c)\,y_2+(c+a)\,x_2\\ x_2 = (ca+ab)/\{2\,(bc+ca+ab)\},\ \&c.\end{aligned}\right\}\ldots(10);$$

(x_3) is given by

$$\left.\begin{aligned}2ax_3-(a+b)\,y_3-(a+c)\,z_3 &= -(a+b)\,x_3+2by_3-(b+c)\,z_3\\ &= -(a+c)\,x_3-(b+c)\,y_3+2cz_3\\ x_3 = (2bc+ca+ab)/\{4\,(bc+ca+ab)\},\ &\&c.\end{aligned}\right\}\ldots\ldots(11).$$

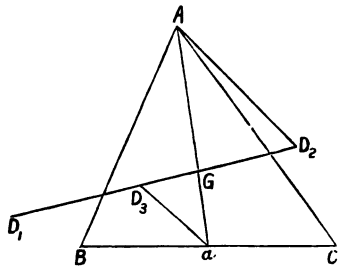
Comparing (9), (10), and (11), we see $x_3 = \frac{1}{2}(x_1+x_2)$, &c.; *i.e.*, the centre of Δ_3 bisects the distance between the centres of Δ_1 and Δ_2. Any point on the line of centres must have coordinates

$$x = \{2nbc+(1-n)(ca+ab)\}/\{2\,(bc+ca+ab)\}.$$

Make $n = \frac{1}{3}$, and we get

$$x = y = z = \tfrac{1}{3},$$

the coordinates of the centroid. That is, the centroid lies on the line of centres $\frac{1}{3}$ of the distance from the centre of Δ_2 towards the centre of Δ_1. Let G be the centroid; D_1, D_2, D_3 the centres of Δ_1, Δ_2, Δ_3; and a the mid-point of BC. Then



$D_2G = 2D_3G$, $AG = 2aG$,

and $\angle AGD_2 = \angle aGD_3$;

therefore triangles AGD_2, aGD_3 are similar, $AD_2 = 2aD_3$, and these lines are parallel. Now a is a point on Δ_3, and A a point on Δ_2. Thus the distances of either G or D_1 from the centres of Δ_2 and Δ_3 are proportional to parallel diameters of these conics. As the conics are similar and similarly situated, it follows that G and D_1 are centres of similitude of Δ_2 and Δ_3.

(v.) The maximum inscribed ellipse in ABC is

$$\Sigma x^2-2\Sigma yz = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(12).$$

The line U will cut this ellipse in real, coincident, or imaginary points as $bc+ca+ab$ is negative, zero, or positive. Again, the line at infinity $(x+y+z=0)$ will cut the conic Δ_1 $(ax^2+by^2+cz^2=0)$ in real, coincident, or imaginary points as $bc+ca+ab$ is negative, zero, or positive.

Thus Δ_1 will be a hyperbola, a parabola, or an ellipse according as U cuts in real, coincident, or imaginary points the maximum inscribed ellipse in ABC. The same follows for Δ_2 and Δ_3 because they are similar to Δ_1.

Passing to ordinary trilinear coordinates, the equation to Δ_1 is still of the form $ax^2+by^2+cz^2=0$; but the coordinates of the centroid are given by $x : y : z = \operatorname{cosec} A : \operatorname{cosec} B : \operatorname{cosec} C$, and U is

$$\Sigma a \operatorname{cosec} A\, x = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

We can pass now to rectangular Cartesian coordinates by a substitution

$$x = \xi\cos\theta_1+\eta\sin\theta_1-p_1,\ y = \xi\cos\theta_2+\eta\sin\theta_2-p_2,\ z = \xi\cos\theta_3+\eta\sin\theta_3-p_3,$$

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where $\theta_3 - \theta_2 = 180° - A$, &c.,

$$\Sigma a x^2 \equiv (\Sigma a \cos^2\theta_1)\,\xi^2 + (\Sigma a \sin^2\theta_1)\,\eta^2 + 2(\Sigma a \sin\theta_1 \cos\theta_1)\,\xi\eta + \ldots$$
$$\equiv A'\xi^2 + B'\eta^2 + 2H'\xi\eta + \ldots, \text{ say} \quad \ldots\ldots(14).$$

The conditions that Δ_1 be a circle are

$$A' - B' = 0, \quad H' = 0, \quad \Sigma a \cos 2\theta_1 = 0, \quad \Sigma a \sin 2\theta_1 = 0,$$

whence $a : b : c = \sin 2A : \sin 2B : \sin 2C$,

which, with (13), gives $\Sigma x \cos A = 0$.

The condition that Δ_1 be a rectangular hyperbola is

$$A' + B' = 0, \text{ } i.e., \quad a + b + c = 0, \quad \Sigma a \sin A \operatorname{cosec} A = 0,$$

which is the condition that U pass through the point

$$x : y : z = \sin A : \sin B : \sin C,$$

i.e., through the symmedian point.

Let $\alpha\xi'^2 + \beta\eta'^2 = 0$ be the equation got from $A'\xi^2 + B'\eta^2 + 2H'\xi\eta + \ldots = 0$ by referring the conic to its principal axes; then

$$\left.\begin{aligned} \alpha + \beta &= A' + B' = a + b + c \\ \alpha\beta &= A'B' - H'^2 = \Sigma bc \sin^2 A \end{aligned}\right\}, \text{ by (14).}$$

The maximum inscribed ellipse, in trilinear coordinates, is

$$\Sigma x^2 \sin^2 A - 2\Sigma yz \sin B \sin C = 0.$$

The pole of $\Sigma x \cos A = 0$ with regard to this ellipse is given by

$$\left.\begin{aligned} x\sin^2 A - y \sin A \sin B - z \sin A \sin C &= \rho \cos A \\ -x \sin B \sin A + y \sin^2 B - z \sin B \sin C &= \rho \cos B \\ -x \sin C \sin A - y \sin C \sin B + z \sin^2 C &= \rho \cos C \end{aligned}\right\},$$

which give $x : y : z = \sin A : \sin B : \sin C$;

i.e., it is the symmedian point.

If the line $lx + my + nz = 0$ pass through this point, $\Sigma l \sin A = 0$; if it touch the maximum inscribed ellipse, $\Sigma \sin A\, mn = 0$. Any conic having double contact with the maximum inscribed ellipse on the line $\Sigma x \cos A = 0$ must have a line characteristic $L\{\Sigma l \sin A\}^2 + M\{\Sigma \sin A\, mn\} = 0$.

If U touch the ellipse,

$$L\{\Sigma a\}^2 + M\{\Sigma bc\,(\sin A/\sin B \sin C)\} = 0,$$
$$L\{a + \beta\}^2 + M(\alpha\beta/\sin A \sin B \sin C) = 0.$$

A given eccentricity gives a constant value to the ratio L : M. That is, as long as Δ_1, Δ_2, Δ_3 have constant eccentricity, U will always touch a conic having double contact with the maximum inscribed ellipse on the line $\Sigma x \cos A = 0$.

(vi.) In what follows areal coordinates are used. The centre of a circumconic

$$lyz + mzx + nxy = 0 \quad \ldots\ldots(6)$$

was found to be given by

$$x : y : z = l(m + n - l) : m(n + l - m) : n(l + m - n) \quad \ldots\ldots(15).$$

The centre of Δ_1 is given by $x : y : z = 1/a : 1/b : 1/c$. If this point lie on (6),

$$al + bm + cn = 0 \quad \ldots\ldots(16).$$

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Substituting the values (15) in the equation to Δ_3, we get

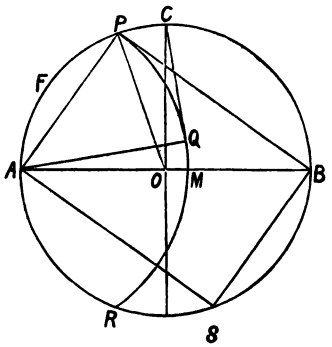
$$\Sigma(b+c)mn(l^2-m^2-n^2+2mn)-\Sigma al^2(m^2+n^2+l^2+2mn-2nl-2lm)$$
$$\equiv lmn\Sigma(b+c-2a)l+\Sigma al^3(m+n)-\Sigma al(m^3+n^3)+\Sigma(b+c)m^2n^2-\Sigma al^4$$
$$\equiv-(\Sigma al)(l^3+m^3+n^3+2lmn-l^2m-l^2n-m^2n-mn^2-lm^2-ln^2)=0,$$

which is satisfied by the condition (16). Hence Δ_3 is the locus of the centres of all circum-conics which pass through the centre of Δ_1.

Since L′M′N′ is the triangle formed by the tangents at A, B, and C, it may be proved, as in (ii.) above, that AL′, BM′, CN′ intersect in the point given by $x:y:z=l:m:n$. Using (16), this gives $ax+by+cz=0$. That is, the point of intersection lies on U.

14565. (Salutation.)—AB, CD are diameters, perpendicular to each other, of the circle ACBD; P is a point on the circumference between A and C; and, with centre A and radius AP, the arc PQR is described, meeting the circle again in R, and having CQ tangential to it. Prove that the arc PQR bisects the circle, when the radius of the latter bears to that portion of its circumference which lies between P and B the same ratio that subsists between CQ^2 and the area (of the circle) lying outside the rectangle APBS.

Solution by F. H. Peachell, B.A.



Let AB = 2r, AP = ρ; then
$$PB=(4r^2-\rho^2)^{\frac{1}{2}}.$$
Let $\angle PBA=\theta$; then
$\angle PAO=\frac{1}{2}\pi-\theta$ and $\angle POA=2\theta$.
Then $\sin\theta=\rho/2r$,
$$\cos\theta=(4r^2-\rho^2)^{\frac{1}{2}}/2r,$$
and $2r^2\sin 2\theta=\rho(4r^2-\rho^2)^{\frac{1}{2}}$.
The arc $AP=2r\theta$, and therefore
$$\text{arc } PB=r\pi-2r\theta.$$
Also $CQ^2=CA^2-AQ^2$
$$=2r^2-\rho^2.$$
The area of rectangle APBS
$$=\rho(4r^2-\rho^2)^{\frac{1}{2}}.$$
The area of part of circle outside this is $\pi r^2-\rho(4r^2-\rho^2)^{\frac{1}{2}}$.
The area cut off by PQR
= twice segment AFP + twice sector PAM
$$=r^2(2\theta-\sin 2\theta)+\rho^2(\tfrac{1}{2}\pi-\theta)=2r^2\theta-\tfrac{1}{2}\rho(4r^2-\rho^2)^{\frac{1}{2}}+\tfrac{1}{2}(\pi\rho^2)-\rho^2\theta.$$
If this is half the circle, that is $=\frac{1}{2}\pi r^2$, we get
$$\pi r^2=4r^2\theta-\rho(4r^2-\rho^2)^{\frac{1}{2}}+\pi\rho^2-2\rho^2\theta,$$

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or $$2\pi r^2 - \pi\rho^2 - 4r^2\theta + 2\rho^2\theta = \pi r^2 - \rho(4r^2-\rho^2)^{\frac{1}{2}},$$

or $$(\pi - 2\theta)(2r^2-\rho^2) = \pi r^2 - \rho(4r^2-\rho^2)^{\frac{1}{2}},$$

or $$(2r^2-\rho^2)/\{\pi r^2 - \rho(4r^2-\rho^2)\}^{\frac{1}{2}} = (\pi - 2\theta)/1 = (\pi r - 2r\theta)/r,$$

or CQ²/(area of circle outside PBSA) = (arc PB)/(radius of the circle).

14597. (Professor Neuberg.)—Trouver les relations entre les abscisses de quatre points pris sur l'hyperbole équilatère $xy = 1$, et tels que les tangentes en ces points touchent une même circonférence.

Solution by Professor Jan de Vries, Ph.D.; *and* Professor Sanjána.

The line $ux + vy + 1 = 0$ will be tangent to the circle

$$(x-a)^2 + (y-b)^2 = c^2, \quad \text{if } au + bv + 1 = \pm c\sqrt{(u^2+v^2)}.$$

Therefore the tangent $y_1x + x_1y = 2$ at the point (x_1, y_1) of the hyperbola $xy = 1$ touches that circle if $ay_1 + bx_1 - 2 = \pm c\sqrt{(x_1^2+y_1^2)}$.

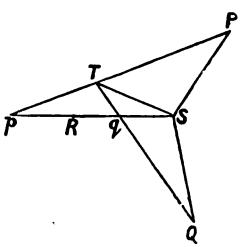
Eliminating a, b, c from this equation and the three analogous equations belonging to the points (x_2, y_2), (x_3, y_3), (x_4, y_4), we obtain the relation

$$\begin{vmatrix} \pm\sqrt{(x_1^2+y_1^2)}, & x_1, & y_1, & 1 \\ \pm\sqrt{(x_2^2+y_2^2)}, & x_2, & y_2, & 1 \\ \pm\sqrt{(x_3^2+y_3^2)}, & x_3, & y_3, & 1 \\ \pm\sqrt{(x_4^2+y_4^2)}, & x_4, & y_4, & 1 \end{vmatrix} = 0.$$

14638. (Rev. T. Roach, M.A.)—Let TP be a tangent to a parabola meeting the axis in p. Prove the following construction for the second tangent:—Centre S, radius ST, describe circle cutting Sp in R, and let

$$Sp : SR = SR : Sq;$$

then Tq is the second tangent.



Solution by I. Nisner; Constance I. Marks, B.A.; *and others.*

Since $Sp : SR = SR : Sq$, $ST^2 = Sp \cdot Sq$; and therefore ST is a tangent to the circle circumscribing the triangle Tpq. Hence $\angle STq = PpS$; whence it follows from Proposition x., on the "Parabola," in C. Taylor's *Elementary Geometry of Conics*, that TQ is the second tangent to the parabola from T.

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14656. (Professor K. J. SANJÁNA, M.A.)—(1) If a, b, c be positive quantities, every two of which are together greater than the third, prove that $\Sigma a^3 - \Sigma a^2 b + 3abc$ is never negative. Hence show that in a plane triangle the inradius is never greater than half the circumradius. (2) With the ordinary notation for a plane triangle, prove that

$$\Sigma a^2 b - \Sigma a^3 = 8S\,(R + r).$$

Solution by JAMES BLAIKIE, M.A.; Rev. T. WIGGINS, B.A.; *and others.*

Since $a + b > c$, $a^2 - b^2 > ac - bc$, therefore $a^3 - ab^2 > a^2 c - abc$. Adding similar expressions, we get $a^3 + b^3 + c^3 - 2\,(a^2 c + b^2 a + c^2 b) + 3abc > 0$.

Similarly, $a^3 + b^3 + c^3 - 2\,(a^2 b + b^2 c + c^2 a) + 3abc > 0$;

therefore $a^3 + b^3 + c^3 - (a^2 b + a^2 c + b^2 a + b^2 c + c^2 a + c^2 b) + 3abc > 0$,

or $$\Sigma a^3 - \Sigma a^2 b + 3abc > 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Since $R = abc/4S$ and $r = S/s$, $R > 2r$, if $abc > 8S^2/s$, that is, if

$$abc > (b + c - a)(c + a - b)(a + b - c) \text{ or } abc > \Sigma a^2 b - \Sigma a^3 - 2abc,$$

which is identical with (1). Also

$$\Sigma a^2 b - \Sigma a^3 = (b + c - a)(c + a - b)(a + b - c) + 2abc$$
$$= 8S\,(S/s + abc/4S) = 8S\,(R + r).$$

14607. (Prebendary WHITWORTH, M.A.)—"Find two unequal integers such that the sum of their reciprocals shall be equal to the reciprocal of a given integer N."—Show that there are

$$\tfrac{1}{2}(2\alpha + 1)(2\beta + 1)(2\gamma + 1) \ldots - \tfrac{1}{2}$$

solutions of this question, where the prime factors of N are given by

$$N = a^\alpha b^\beta c^\gamma \ldots .$$

For example, if N = 12, there are seven solutions, viz.,

$$\tfrac{1}{12} = \tfrac{1}{13} + \tfrac{1}{156} = \tfrac{1}{14} + \tfrac{1}{84} = \tfrac{1}{15} + \tfrac{1}{60} = \tfrac{1}{16} + \tfrac{1}{48} = \tfrac{1}{18} + \tfrac{1}{36}$$
$$= \tfrac{1}{20} + \tfrac{1}{30} = \tfrac{1}{21} + \tfrac{1}{28}.$$

Solutions by Professor JAN DE VRIES, Ph.D.; H. W. CURJEL, M.A.; *and many others.*

The equation $1/x + 1/y = 1/N$ may be put into the form

$$(x - N)(y - N) = N^2.$$

The number of divisors of N^2 being represented by

$$(2\alpha + 1)(2\beta + 1)(2\gamma + 1) \ldots,$$

there will be $\{(2\alpha + 1)(2\beta + 1)(2\gamma + 1) \ldots - 1\} : 2$ different solutions. For the solution $x - N = y - N = N$ must be excluded.

14669. (V. DANIEL.)—Triangles are inscribed one to another so that the sides of one are divided, in the same sense, by the vertices of the next

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in a constant ratio $k/(1-k)$. Show that the sides of the nth inscribed triangle, in terms of a, b, c (the sides of the outermost), are given by

$$-3a_n^2 = 4\lambda^n (a^2 \cos\theta_1 \cos\theta_2 + b^2 \cos\theta_2 \cos\theta_3 + c^2 \cos\theta_3 \cos\theta_1),$$

and two similar expressions with a, b, c, changed in cyclic order, where $\cos\theta_1$, $\cos\theta_2$, $\cos\theta_3$ are the roots of the trigonometrical cubic

$$4\cos^3\theta - 3\cos\theta + \sin 3n\phi = 0,$$

and $$\phi = -\tfrac{1}{6}\pi + \sin^{-1}\tfrac{1}{2}\lambda^{\frac{1}{2}} = \tfrac{1}{3}\cos^{-1}(\tfrac{3}{2}\lambda^{-\frac{1}{2}} - \tfrac{1}{2}\lambda^{-\frac{3}{2}}),$$

and $$\lambda = 1 - 3k + 3k^2.$$

Solution by the Proposer.

Let A′B′C′ be the first inscribed triangle, A″B″C″ the second, and so on. Let the Cartesian coordinates of the A vertices be denoted by x_A, x'_A, x''_A, ..., y_A, y'_A, y''_A, ... Similarly for the B and C vertices.

Then
$$x'_A = (1-k)\,x_A + k\,.\,x_B,$$
$$x'_B = (1-k)\,x_B + k\,.\,x_C,$$
$$x'_C = (1-k)\,x_C + k\,.\,x_A,$$
$$x''_A = (1-k)\,x'_A + k\,.\,x'_B = (1-k)^2 x_A + 2k\,(1-k)\,x_B + k^2\,.\,x_C,$$
$$x''_B = \quad\quad = (1-k)^2 x_B + 2k\,(1-k)\,x_C + k^2\,.\,x_A,$$

and it is evident, from the manner of formation, that

$$x_A^{(n)} = (1-k)^n x_A + {}^nC_1\,(1-k)^{n-1}\,k\,.\,x_B + {}^nC_2\,(1-k)^{n-2}\,k^2\,.\,x_C + \ldots$$

to $n+1$ terms

$$\left.\begin{aligned} &= p_n\,.\,x_A + q_n\,.\,x_B + r_n\,.\,x_C \\ x_B^{(n)} &= p_n\,.\,x_B + q_n\,.\,x_C + r_n\,.\,x_A \\ x_C^{(n)} &= p_n\,.\,x_C + q_n\,.\,x_A + r_n\,.\,x_B \end{aligned}\right\} \text{ with similar expressions for the ordinates,}$$

where the coefficients p_n, q_n, r_n are the three sums formed by taking every third term in the binomial expansion of $(1-k+k)^n$, beginning with the first three terms. We have, using ω and ω^2, to denote the imaginary cube roots of unity:

$$(1-k+k)^n = (1-k)^n + {}^nC_1\,(1-k)^{n-1}\,k + {}^nC_2\,(1-k)^{n-2}\,k^2 + \ldots$$
$$\ldots + {}^nC_{n-1}\,(1-k)\,k^{n-1} + k^n,$$
$$(1-k+\omega k)^n = (1-k)^n + {}^nC_1\,(1-k)^{n-1}\,k\omega + {}^nC_2\,(1-k)^{n-2}\,k^2\omega^2 + \ldots$$
$$\ldots + {}^nC_{n-1}\,(1-k)\,k^{n-1}\,\omega^{n-1} + k^n\omega^n$$
$$(1-k+\omega^2 k)^n = (1-k)^n + {}^nC_1\,(1-k)^{n-1}\,k\omega^2 + {}^nC_2\,(1-k)^{n-2}\,k^2\omega^4 + \ldots$$
$$\ldots + {}^nC_{n-1}\,(1-k)\,k^{n-1}\,\omega^{2n-2} + k^n\omega^{2n}.$$

Hence, since $1 + \omega^m + \omega^{2m} = 0$ or 3, we get

$$3p_n = 1 + (1-k+\omega k)^n + (1-k+\omega^2 k)^n,$$
$$3q_n = 1 + \omega^2\,(1-k+\omega k)^n + \omega\,(1-k+\omega^2 k)^n,$$
$$3r_n = 1 + \omega\,(1-k+\omega k)^n + \omega^2\,(1-k+\omega^2 k)^n.$$

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Using De Moivre's theorem, it follows easily that

$$3(p_n - q_n) = 2\sqrt{3}\,\lambda^{\frac{1}{3}n}\cos(n\theta + \tfrac{1}{6}\pi), \quad 3(q_n - r_n) = 2\sqrt{3}\,\lambda^{\frac{1}{3}n}\cos(n\theta + \tfrac{5}{6}\pi),$$
$$3(r_n - p_n) = 2\sqrt{3}\,\lambda^{\frac{1}{3}n}\cos(n\theta + \tfrac{3}{2}\pi),$$

where $\theta = \tan^{-1} k\sqrt{3}/(2-3k) = -\frac{1}{6}\pi + \sin^{-1} 1/2\sqrt{\lambda}$ and $\lambda = 1 - 3k + 3k^2$.

Taking the side c_n of the nth triangle inscribed,

$$\begin{aligned}
c_n^2 &= (x_A^{(n)} - x_B^{(n)})^2 + (y_A^{(n)} - y_B^{(n)})^2 \\
&= \{p_n(x_A - x_B) + q_n(x_B - x_C) + r_n(x_C - x_A)\}^2 \\
&\qquad + \{p_n(y_A - y_B) + q_n(y_B - y_C) + r_n(y_C - y_A)\}^2 \\
&= p_n^2 c^2 + q_n^2 a^2 + r_n^2 b^2 + 2 \sum_{pqr,\,\Delta ABC} p_n q_n \{(x_A - x_B)(x_B - x_C) \\
&\qquad + (y_A - y_B)(y_B - y_C)\} \\
&= p_n^2 c^2 + q_n^2 a^2 + r_n^2 b^2 - 2\Sigma ac \cos B\, p_n q_n \\
&= p_n^2 c^2 + q_n^2 a^2 + r_n^2 b^2 - p_n q_n (c^2 + a^2 - b^2) - q_n r_n (a^2 + b^2 - c^2) \\
&\qquad - r_n p_n (b^2 + c^2 - a^2) \\
&= -\Sigma (p_n - q_n)(q_n - r_n)\, a^2 \\
&= -\tfrac{1}{4}\lambda^n (a^2 \cos\theta_1 \cos\theta_2 + b^2 \cos\theta_2 \cos\theta_3 + c^2 \cos\theta_3 \cos\theta_1),
\end{aligned}$$

by substituting the values of the products $(p_n - q_n)(q_n - r_n)$, &c., and it is evident the given cubic is satisfied by θ_1, θ_2, θ_3.

14485. (Professor Sanjána, M.A.) — Eliminate p and q from $x = p(p - 2^{\frac{1}{2}})$, $y = q(p + 2^{\frac{1}{2}})$, and $p^2 - q^2 = 4$.

Solution by R. Chartres; Constance Isabelle Marks, B.A.; *and* Lt.-Col. Allan Cunningham, R.E.

Eliminating q from the second and third equations, and substituting for p in the first, we get, after reduction,

$$8(x-1)(y^2 + 3x^2 - 8x + 8) = (y^2 - x^2 - 4x + 8)^2.$$

14387. (Lt.-Col. Allan Cunningham, R.E.)—Show how to find a sum of successive cubes equal to q times a square (where q is an integer containing no square factor). Give solutions when $q = 2, 3, 5$, &c.

Solution by the Proposer.

Let $S_n = 1^3 + 2^3 + 3^3 + \ldots + n^3$, $T_m = \frac{1}{2}m(m+1)$,

let $S_n = 1^3 + 2^3 + 3^3 + \ldots + n^3$, $T_n = \frac{1}{2}n(n+1)$,

let $S_{m,n} = (n+1)^3 + (n+2)^3 + \ldots + (m-1)^3 + m^3$ = required sum.

Then $$S_{m,n} = S_m - S_n = T_m^2 - T_n^2 = qz^2.$$

Assume $T_m = \xi \,.\, T_n$, when ξ = *integer* (an assumption to be justified by the result). This gives $z = \zeta \,.\, T_n$, where ζ = *integer* and $\xi^2 - q \,.\, \zeta^2 = 1$.

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Every solution (ξ, ζ) of this diophantine will be a solution of the question, provided that $T_m = \xi . T_n$ is also satisfied. This last can now be readily solved (with the values of ξ just found), by aid of a table of triangular numbers (*e.g.*, DE JONCOURT'S, 1772).

Thus, multiply the lower numbers (T_n) of the table by the (given) ξ, until a product $\xi . T_n$ is found = another triangular (T_m); the table then gives m, n; and z is given by $z = \zeta . T_n$.

Or, again, the condition $T_m = \xi . T_n$ may be solved without tables; for it gives $m^2 + m = \xi (n^2 + n)$; and, writing $x = 2m + 1$, $y = 2n + 1$, this gives $\xi . y^2 - x^2 = (\xi - 1)$, where ξ is *given*. Every solution of this in *odd* numbers (x, y) solves the Question.

[Note that this also solves the problem of finding a sum of cubes of the successive *multiples* of a given number q (where q contains no square factor); for here $S'_m = q^3 . T^2_m$, $S'_n = q^3 . T^2_n$.]

Examples.—The following table shows the elements (ξ, ζ, T_m, T_n) of several solutions together with the final quantities m, n, z completing the solutions for each of the small values of $q = 2, 3, 5, 6, 7, 10, 11$:—

	$q = 2$		
	$\xi = 2, \zeta = 2$	$\xi = 17, \zeta = 12$	$\xi = 99, \zeta = 70$
$T_m =$	3, 45, 630, 8778	1326, 3570	990, 1485
$T_n =$	1, 15, 210, 2926	78, 210	10, 15
$m =$	2, 9, 35, 132	51, 84	44, 54
$n =$	1, 5, 20, 76	12, 20	4, 5
$z =$	2, 30, 420, 5852	936, 2520	700, 1050

	$q = 3$			
	$\xi = 2, \zeta = 1$	$\xi = 7, \zeta = 4$	$\xi = 26, \zeta = 15$	$\xi = 97, \zeta = 56$
$T_m =$	6, 210, 7140	21, 105, 2415	78, 5460	42195
$T_n =$	3, 105, 3570	3, 15, 325	3, 210	435
$m =$	3, 20, 119	6, 14, 69	12, 104	90
$n =$	2, 14, 84	2, 5, 25	2, 20	29
$z =$	3, 105, 3570	12, 60, 1300	45, 3150	24360

	$q = 5$	$q = 6$	$q = 7$	$q = 10$	$q = 11$
	$\xi = 161, \zeta = 72$	$\xi = 5, \zeta = 2$	$\xi = 8, \zeta = 3$	$\xi = 19, \zeta = 6$	$\xi = 10, \zeta = 3$
$T_m =$	2415	15, 105	120, 528	190	10, 210, 780
$T_n =$	15	3, 21	15, 66	10	1, 21, 78
$m =$	69	5, 14	15, 32	19	4, 20, 39
$n =$	5	2, 6	5, 11	4	1, 6, 12
$z =$	1080	6, 42	45, 196	60	3, 63, 224

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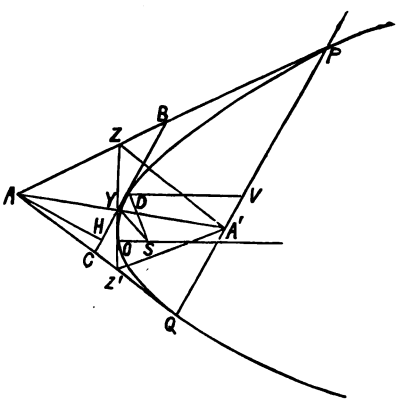
14198. (Professor SANJÁNA, M.A.)—If p_1, p_2, p_3 be the parameters of the three parabolas which respectively touch one side of a triangle at its mid-point and the other two sides produced, then

$$\Sigma p^{-\frac{1}{3}} = \tfrac{3}{2}\frac{\cot\omega}{(2\Delta)^{\frac{1}{3}}} \quad \text{and} \quad \Sigma p^{-\frac{1}{3}} = \tfrac{9}{8}\frac{\cot^2\omega - 1}{(2\Delta)^{\frac{1}{3}}}.$$

[See Quest. 13002, Vol. LXV.]

Solution by the PROPOSER.

Let two tangents AP, AQ be drawn to a parabola whose vertex is O; let the tangent at the vertex meet them in Z, Z′; complete the parallelogram AZA′Z′. Then A′ is on PQ (MILNE, § 33). Now AA′, ZZ′ bisect each other at Y, and the diameter ADV is bisected at D; hence YD is parallel to PQ; *i.e.*, touches the curve at D. Let it meet the tangents from A in B, C; then the parabola touches BC at its mid-point, and also AB and AC produced. Let AH be perpendicular to BC, and let d denote the circumdiameter of ABC.



Now $\frac{AH}{AB} = \frac{AC}{d}$ (Euc. VI., c), and, as the circle ABC passes through S, $\frac{SY}{SB} = \frac{SC}{d}$ also. The triangles SYO, ADH are similar; so also are SAB, ADC, and SAC, ADB; hence

$$\frac{AH}{AD} = \frac{SO}{SY}, \quad \frac{SB}{AB} = \tfrac{1}{2}\frac{BC}{AD}, \quad \frac{SC}{AC} = \tfrac{1}{2}\frac{BC}{AD}.$$

Compounding these ratios, we get

$$\frac{SO \,.\, AD}{AH^2} = \tfrac{1}{4}\frac{BC^2}{AD^2} = \frac{\Delta^2}{AD^2 \,.\, AH^2};$$

so that $SO = \frac{\Delta^2}{m_1{}^3}$, where Δ is the area of ABC and m_1 is the median from A. [This process solves Quest. 13002 geometrically.]

It is obvious that $p_1 = \frac{4\Delta^2}{m_1{}^3}$; similarly $p_2 = \frac{4\Delta^2}{m_2{}^3}$, $p_3 = \frac{4\Delta^2}{m_3{}^3}$. Hence

$$\Sigma p^{-\frac{1}{3}} = \Sigma\frac{m^2}{2^{\frac{1}{3}}\Delta^{\frac{1}{3}}} = \frac{3(a^2+b^2+c^2)}{4\,.\,2^{\frac{1}{3}}\Delta^{\frac{1}{3}}} = \tfrac{3}{2}\frac{\cot\omega}{(2\Delta)^{\frac{1}{3}}};$$

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and $\Sigma p^{-\frac{4}{3}} = \Sigma \frac{m^4}{2^{\frac{4}{3}} \Delta^{\frac{4}{3}}} = \frac{9(a^4+b^4+c^4)}{16 \cdot 2^{\frac{4}{3}} \Delta^{\frac{4}{3}}} = \frac{9\{(4\Delta \cot \omega)^2 - 2(4\Delta^2 \operatorname{cosec}^2 \omega)\}}{16 \cdot 2^{\frac{4}{3}} \Delta^{\frac{4}{3}}}$

$$= \frac{9}{8} \frac{\cot^2 \omega - 1}{(2\Delta)^{\frac{4}{3}}}.$$

14614. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Factorize completely $(1792^7 + 1)$.

I. *Solution by the* PROPOSER.

Let $N = N_1 N_2$, where $N_1 = (1792+1) = 11 \cdot 163$, $N_2 = \frac{1792^7+1}{1792+1}$.

Now, every number N_2 of form $(x^7+y^7) \div (x+y)$ is resolvable into two factors, viz., $N_2 = (P-Q)(P+Q)$, where

$$P = (x+y)^3, \quad Q = \sqrt{(7xy)}\,(x^2+xy+y^2),$$

by the *Aurifeuillian* resolution, see ED. LUCAS's *Sur la Série récurrente de Fermat* (Rome, 1879), p. 6, and, when—as in this case—$x = 7\xi^2$, $y = \eta^2$, then $\sqrt{(7xy)} = 7\xi\eta$, and the factors are *both real.*

Here $x = 1792 = 7 \cdot 16^2$, $y = 1$; $\xi = 16$, $\eta = 1$;

therefore $N_2 = (P-Q)(P+Q) = 5404361873 \cdot 6124086641$.

Next, the PROPOSER has computed a MS. table (which it is hoped shortly to publish) showing *all* the roots (x) of $(x^7 \mp 1) \div (x \mp 1) \equiv 0$, (mod p), for all primes $(p) \not> 10000$. This gives the divisors 113, 197, 2437; and by actual division into $(P-Q)$, $(P+Q)$, it is found

$$P-Q = 197 \cdot 2437 \cdot 11257; \quad P+Q = 113 \cdot 54195457.$$

The MS. table referred to shows that there are no more prime divisors $\not> 10000$ of N_2, so that the large factor in $(P+Q)$ must be *prime.* Thus $N = N_1 \cdot N_2$ is completely resolved.

II. *Solution by* H. J. WOODALL, A.R.C.S.

$$1792^7 + 1 = 11 \cdot 163 \; ; \; 197 \cdot 2437 \cdot 11257 \; : \; 113 \cdot 54195457 \; ;$$

all but the largest factor may be obtained from certain tables of congruence solutions now in progress, and of which the following is an abstract:—

Prime Modulus.	l = period of 2.	Solution of $2^r \equiv \pm 7$			Test.
		r	ρ	$\pm$	
11	10	2	1	−	$8+2 = 10 = l$
163	162	73	1	+	$7 \times (8+73) = 3\frac{1}{2} \times l$
197	196	48	1	−	$7 \times (8+48) = 2 \times l$
2437	2436	862	1	+	$7 \times (8+862) = 2\frac{1}{2} \times l$
11257	1407	394	1	−	$7 \times (8+394) = 2 \times l$
113	28	10	1	+	$7 \times (8+10) = 4\frac{1}{2} \times l$

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14601. (Professor K. J. SANJÁNA.)—An object is placed in front of a combination of three coaxial convex lenses, whose focal lengths are f_1, f_2, f_3 respectively, and the successive distances between them a, b. If the final image is in the same position as the object, prove that

$$(f_1-a)(f_3-b)+f_2(f_1-a+f_3-b) = 0 \quad \ldots\ldots\ldots\ldots\ldots (1);$$

$$2f_1f_2(a+b)+2f_1f_3a-f_1b(2a+b)-f_2(a+b)^2-f_3a^2+ab(a+b) = 0 \ldots (2);$$

and

$$(a+b)^2f_2+a^2f_3-ab(a+b) = 0 \quad \ldots\ldots\ldots\ldots\ldots (3).$$

If a, b be given, the focal lengths are $a(a+b)/2b$, $ab/2(a+b)$, $b(a+b)/2a$ respectively, and the final image is equal to the object in magnitude. [A particular case is the system formed by three convex lenses of focal lengths $\frac{3}{2}$, $\frac{1}{6}$, $\frac{3}{8}$, at intervals 1, $\frac{1}{2}$; see WOLSTENHOLME, No. 2781.]

Solution by "$\theta.\ \beta.$," Cambridge.

The image of the point at infinity on the axis is to be at infinity; hence both focal points of the combined system are at infinity. The magnification is then constant ($= m$, say), and the relation between the positions of object and image is $x/m+ym = \text{const.}$; if the object and image coincide, this will be of the type $x+y = \text{const.}$, so that $m = \pm 1$. Thus the focal length of the system is infinite (or the power is zero), and the magnification is (± 1) for all points.

Let p, q be the distances between adjoining focal points of the lenses, so that $p = a-(f_1+f_2)$, $q = b-(f_2+f_3)$. Then, in order that the focal length may be infinite, $pq = f_2^2$; and if x be the distance of an object from the first focal point of the first lens, and y that of the image from the second focal point of the last lens, we have

$$(p-f_1^2/x)(q-f_3^2/y) = f_2^2 = pq.$$

Thus

$$px/f_1^2+qy/f_3^2 = 1,$$

and this relation is to be the same as $x+y+f_1+f_3+a+b = 0$.

Hence

$$f_1^2/(a-f_1-f_2) = f_3^2/(b-f_2-f_3) = -(a+b+f_1+f_3),$$

and, remembering $pq = f_2^2$, each of the equal quantities is also $\pm f_1f_3/f_2$. Assuming that f_1, f_2, f_3 are all positive, we must take the lower sign, and then we have the conditions in a symmetrical form. By combination we find $(a-f_1)/f_1 = -(b-f_3)/f_3 = t$, say, and $2(af_3-bf_1)+a^2-b^2 = 0$.

Substituting for f_1, f_3 in terms of t, we get a quadratic giving $t = (a+b)/(a-b)$ or $(b-a)/(b+a)$. The former gives $2f_1 = a-b = -2f_3$; so that (since f_1, f_3 are both positive) we have to take the second root; then we have the values given in the question

$$f_1 = a(a+b)/2b, \quad f_2 = ab/2(a+b), \quad f_3 = b(a+b)/2a.$$

It is not hard to see that in this case the magnification is -1, or the image is equal to the object, but inverted.

The question was set in the Cambridge Mathematical Tripos (1893), the result given there being

$$af_3^{\frac{1}{2}} = bf_1^{\frac{1}{2}} = 2f_1f_3/(f_1^{\frac{1}{2}}+f_3^{\frac{1}{2}});$$

but this is not complete, as f_1, f_2, f_3 are subject to the relation

$$f_1^{-\frac{1}{2}}+f_3^{-\frac{1}{2}} = f_2^{-\frac{1}{2}}.$$

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6835. (Col. A. R. CLARKE, C.B., F.R.S.)—Two equal and similar arcs of a circle, of angular extent α, are drawn on a piece of paper. They intersect: what are the respective chances of one and two intersections?

Solution by D. BIDDLE.

There are at least two cases to this Question, according as the arc is a minor (Figs. 1, 2) or a major one (Fig. 3).

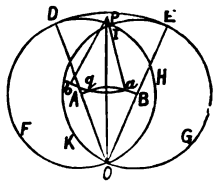

FIG. 1.

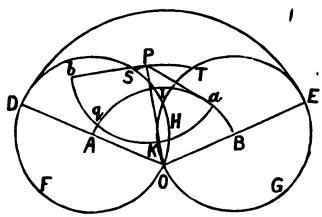

FIG. 2.

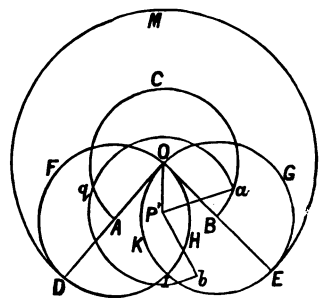

FIG. 3.

Case 1.—Let AB (Fig. 1) be the arc with centre O. From A, B as centres describe circles, with radius AO = BO, intersecting in O, I. Join OA, OB, and produce to D, E. With centre O and radius OD, describe the arc DE. Then it is easy to see that P, the centre of the second arc *ab*, cannot lie in OHIK, the space common to the two circles, because AB and *ab* could not then intersect at all. If P lie in the complementary spaces OFIK, OGIH of the two circles, single intersection alone is possible. In order that double intersection may occur, P must lie in the space IDEI, outside the circles but within the sector ODE. These remarks apply equally to Fig. 2, although I is not on the same side as DE of the arc AB. But it is not sufficient that P be in the space IDEI to ensure that if intersection occur it shall be double. The chance of double intersection, even when P is favourably placed, is represented by $bq/(bq + 2aq)$.

If, now, we refer to Fig. 2, it will be easy to see that this secondary

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chance is the same whilst P remains on SPT, equidistant from O, but within the prescribed space. It is also clear that SPT subtends an equal angle from O that bq does from P. Taking the radius of the given arcs as unity, this angle is $\alpha - 2\cos^{-1}\frac{1}{2}$OP; and OP varies from $2\cos\frac{1}{2}\alpha$ to 2. Therefore the secondary chance mentioned above is given for the whole of IDEI by the following integral,

$$\frac{2}{\alpha-\sin\alpha}\int_{\cos\frac{1}{2}\alpha}^{1}\frac{(\alpha-2\cos^{-1}x)^2x}{\alpha+2\cos^{-1}x}\,dx,$$

where $\alpha-\sin\alpha$ represents the area of IDEI. Each of the spaces OFIK, OGIH equals $\alpha+\sin\alpha$; therefore the total space in which P can lie to ensure intersection is $3\alpha+\sin\alpha$. Consequently the chance of double intersection is given by

$$\frac{2}{3\alpha+\sin\alpha}\int_{\cos\frac{1}{2}\alpha}^{1}\frac{(\alpha-2\cos^{-1}x)^2x}{\alpha+2\cos^{-1}x}\,dx;$$

and the chance of single intersection is the difference between this and 1.

Case 2.—Here the conditions (see Fig. 3) are altered, ACB is the fixed arc, and circles are described from its extremities as centres as before. But now not only is the space DMEGOFD privileged to afford an opportunity for double intersection when P is placed in it, but the space OHIKO as well, which is common to the two circles. The area of DMEGOFD is $4\alpha-\pi$; the area of OHIKO is $\alpha-\pi-\sin(\alpha-\pi)$; the complementary area of each circle is $2\pi-\alpha+\sin(\alpha-\pi)$. Consequently the total area within which P can lie to ensure intersection is $3\alpha+2\pi+\sin(\alpha-\pi)$. Of this $5\alpha-2\pi-\sin(\alpha-\pi)$ affords an opportunity for double intersection, but the estimation of the chance involves two integrals instead of only one:—

$$\frac{2\displaystyle\int_0^1\frac{(\alpha-2\cos^{-1}x)^2x}{\alpha+2\cos^{-1}x}\,dx+2\int_0^{\cos(\pi-\frac{1}{2}\alpha)}\frac{(\alpha-2\cos^{-1}x)(\alpha+2\cos^{-1}x-2\pi)x}{\alpha+2\cos^{-1}x}\,dx}{3\alpha+2\pi+\sin(\alpha-\pi)},$$

and the difference between this and 1 is the chance of single intersection in Case 2.

14585. (J. H. Taylor, M.A.)—If three circles, which touch a fixed straight line at the focus S of a parabola, touch the parabola in P, Q, R, and cut in points p, q, r a circle which, centred on the axis, passes through S and touches the parabola, the circles pSO′, qSO′, rSO′ (where O′ is the reflection of the focus in the directrix) intersect at 60°.

Solution by the Proposer.

This problem is the inverse of Dr. Archibald's Quest. 14495 (Vol. lxxiv., p. 131), and the following is a direct solution:—

$$y^2-4ax=0 \quad \text{.....(A)}$$

is the equation to a parabola.

$$(1+m^2)(y^2-4ax)-(y-mx-a/m)(y+mx-\mathrm{A})=0 \quad \text{.....(B)}$$

represents a circle touching (A) at its tangent and cutting the parabola in a chord at an equal angle in the negative direction.

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If this passes through the focus, $A = -3am$ and (B) becomes

$$m^2(x^2+y^2) - a(3+m^2)x + a(1/m - 3m)y + 3a^2 = 0 \quad \text{.....(B')}.$$

The cube of m shows that there are three such circles. Again,

$$y^2 + (x-5a)^2 = (4a)^2 \quad \text{.....(C)}$$

is the equation of a circle centred on the axis, passing through the focus, and touching the parabola.

$$\lambda\{y^2 + (x-5a)^2 - (4a)^2\} + m^2(x^2+y^2) - a(3+m^2)x + a/m\,(1-3m^2)y + 3a^2 = 0 \quad \text{.....(D)}$$

represents a circle through the intersections of (B') and (C). The condition that this circle passes through O' is $\lambda = -\frac{1}{4}(1+m^2)$.

To find the tangent of the angle ϕ made with the axis of x by the tangent to (D) at the focus, write $(x+a)$ for x, and pick out the terms of the first degree in x and y; this gives $y/x = m = \tan\phi$.

If θ is the angle which the tangent to (B') at the focus makes with the axis of x, in the same way, we find

$$\tan\theta = (m^3 - 3m)/(3m^2 - 1) = \tan 3\phi,$$

whichever value of m be taken; therefore $3\phi = n\pi + \theta$, $3\phi' = n'\pi + \theta$ (where n and n' are integers); therefore $\phi' - \phi = (m-n)\frac{1}{3}\pi$.

[Dr. Archibald, bearing out the opening statement of the above solution, observes:—This theorem follows at once from the theorem of Quest. 14495, by inversion of the cardioid with respect to a circle, of radius equal to the diameter of the cardioid's base, and with centre at the cardioid's real cusp.]

6723. (C. Leudesdorf, M.A.)—A pair of tangents to a given conic form an harmonic pencil with two straight lines whose directions are given and which include a right angle. Show that the locus of the point of intersection of the tangents is a rectangular hyperbola, except in the case where the given conic is a parabola, when the locus is a straight line.

Solution by Prince C. de Polignac.

1. Let the given conic be the ellipse $b^2X^2 + a^2Y^2 = a^2b^2$, and let one of the given directions make an angle θ with the major axis a. Transforming to new rectangular axes parallel to the given directions through the formulæ $X = x\cos\theta - y\sin\theta$, $Y = x\sin\theta + y\cos\theta$, the new equation of the ellipse will be

$$Ax^2 + 2Bxy + Cy^2 - a^2b^2 = 0 \quad \text{.....(1)},$$

where

$$B = (a^2 - b^2)\sin\theta\cos\theta = \tfrac{1}{2}c^2\sin 2\theta,$$

and through a well known invariantive property

$$AC - B^2 = a^2b^2 \quad \text{.....(2)}.$$

Now, since the pair of tangents form a harmonic pencil with the axes, their equations must be

$$y = \mu x + k, \quad y = -\mu x + k' \quad \text{.....(3, 4)}.$$

Combining (1) and (3), we get

$$(A + 2B\mu + C\mu^2)x^2 + 2(B + C\mu)kx + Ck^2 - a^2b^2 = 0,$$

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and the condition of contact is

$$(A + 2B\mu + C\mu^2)(Ck^2 - a^2b^2) - (B + C\mu)^2 k^2 = 0.$$

Expanding, the factor a^2b^2 drops out on account of equation (2), and we obtain, finally, $C\mu^2 + 2B\mu - k^2 + A = 0$ .....(5).

In the same way, $C\mu^2 - 2B\mu - k'^2 + A = 0$ .....(6)

for the second tangent.

The sought locus, which passes through the focus of the given conic, is the result of eliminating μ, μ', k between equations (3), (4), (5), (6).

Equations (3), (4) give at once

$$2y = k + k', \quad 2\mu x = k' - k \quad \ldots\ldots (7, 8);$$

therefore $4\mu xy = k'^2 - k^2$. Subtracting (6) from (5), we get $4B\mu = k^2 - k'^2$, whence for the locus $xy + B = 0$, *i.e.*, $xy = -\frac{1}{2}c^2 \sin 2\theta$, a rectangular hyperbola.

The form of its equation shows that the locus is the same for all ellipses confocal to the given one, a property not stated in the Question.

For $\theta = 0$ the locus reduces to the axes of the ellipse, as is evident *a priori*.

2. The given conic is the hyperbola $x^2/a^2 - y^2/b^2 = 1$. Same conclusions, changing in 1, b^2 to $-b^2$.

3. The given conic is a parabola. Let its equation be $Y^2 = 2pX$. Transforming again to rectangular axes parallel to the given fixed directions, we get $(x \sin\theta + y\cos\theta)^2 = 2p\,(x\cos\theta - y\sin\theta)$.....(9).

Substituting into this equation from (3)

$$\{(\sin\theta + \mu\cos\theta)\,x + k\cos\theta\}^2 + 2p\,\{(\mu\sin\theta - \cos\theta)\,x + k\sin\theta\} = 0,$$

and the condition of contact will be

$$(\sin\theta + \mu\cos\theta)^2\,(k^2\cos^2\theta + 2pk\sin\theta) - \{(\sin\theta + \mu\cos\theta)\,k\cos\theta + p\,(\mu\sin\theta - \cos\theta)\}^2.$$

The term $(\sin\theta + \mu\cos\theta)^2 k^2\cos^2\theta$ disappears, and the remaining terms can be written, noting that the factor p drops out,

$$2k\,(\mu\cos\theta + \sin\theta)\,\{(\mu\cos\theta + \sin\theta)\sin\theta - (\mu\sin\theta - \cos\theta)\cos\theta\} - p\,(\mu\sin\theta - \cos\theta)^2 = 0,$$

i.e., $2k\,(\mu\cos\theta + \sin\theta) - p\,(\mu\sin\theta - \cos\theta)^2 = 0$ .....(10).

Changing μ to $-\mu$, we obtain for the other tangent (4) the condition

$$2k'\,(\sin\theta - \mu\cos\theta) - p\,(\mu\sin\theta + \cos\theta)^2 = 0 \ldots\ldots(11).$$

We have here to eliminate $\mu kk'$ between (10), (11), (3), (4), or, which is the same, (10), (11), (7), (8). We find, (10), (11),

$$(k + k')\,\mu\cos\theta + (k - k')\sin\theta + 2p\mu\sin\theta\cos\theta = 0,$$

or $$2\mu y\cos\theta - 2\mu x\sin\theta + 2p\mu\sin\theta\cos\theta = 0,$$

giving for the locus the right line

$$y = x\tan\theta - p\sin\theta \ldots\ldots(12),$$

which makes with the axis of the given parabola the angle 2θ.

Transforming back to the original axes, the equation of the right line is

$$Y = \{X - \tfrac{1}{2}P\}\tan 2\theta \ldots\ldots(13).$$

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Here again it can be shown that the locus will be the same for all confocal and coaxal parabolas. The general equation of parabolas confocal and coaxal with $Y^2 = 2pX$ is $Y^2 = 2(p+2a)\{X+a\}$, where a is an arbitrary parameter. Place $p+2a = P$ and $X+a = (X)$, which amounts to a change of origin on the X axis. The equation of the confocal and coaxal parabolas is then $Y^2 = 2P(X)$.....(14),

and consequently the locus of the point of intersection of the tangents to (14) which form a harmonic pencil with two given rectangular directions one of which makes an angle θ with the axis of the parabola is, from (13), $Y = \{(X)-\frac{1}{2}P\}\tan 2\theta$ to the axes (X), Y; and to the axes X, Y,

$$Y = \{X+a-\tfrac{1}{2}P\}\tan 2\theta, \quad i.e., \quad Y = \{X-\tfrac{1}{2}P\}\tan 2\theta,$$

which is equation (13).

[For another solution of this Question see p. 58 of the present volume.]

14553. (R. Tucker, M.A.)—AP, AQ are vertical vectors of a parabola, and PQ is a focal chord. The circle APQ cuts the curve in R. The conormal point (of PQR) lies on a parabola, as also does the circumcentre of APQ. The intersections of (AP, QR) and of (AQ, PR) lie on the same cubic, and PR touches a semi-cubical parabola. AR, PQ intersect on a parallel to the directrix.

Solution by F. H. Peachell, B.A.; *and* W. J. Greenstreet, M.A.

Let coordinates of P be am^2, $2am$. Then coordinates of Q are a/m^2, $-2a/m$. Now, if a circle cut a parabola in four points, the sum of the ordinates of the points is equal to zero. The points A, P, Q, R are concyclic; therefore ordinate of R $= -2am + 2a/m - 0 = 2a(1-m^2)/m$, and abscissa $= a(1-m^2)^2/m^2$.

(1) Let coordinates of conormal point be h, k. Any normal through h, k is given by $(k-y)\,2a + y(h-x) = 0$,

or $4a^2k^2 = y^2(2a-h+x)^2$, or $4a^2k^2 = 4ax(2a-h+x)^2$.

If x_1, x_2, x_3 be abscissæ of the points the normals at which meet in h, k, then

$$2(h-2a) = x_1+x_2+x_3 = am^2 + a/m^2 + a/m^2(1-2m^2+m^4)$$
$$= 2am^2 + 2a/m^2 - 2a;$$

therefore $h = am^2 + a/m^2 + a$, and k is found to be $a(m-1/m)$. Eliminating m, we get locus $k^2 = a(h-3a)$, a parabola.

(2) Locus of circumcentre of APQ. If x, y be coordinates, we find

$$y = \tfrac{1}{4}a(m-1/m), \quad x = 2a + \tfrac{1}{2}a(m^2-1+1/m^2).$$

Eliminating m, we get locus of the centre $16y^2 = a(2x-5a)$, a parabola.

(3) QR joins a/m^2, $-2a/m$, and $a(1-m^2)^2/m^2$, $2a(1-m^2)/m$. Equation of QR is $m^3y + 2m^2x - 2a + 2am^2 = 0$.....(1),

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and equation of AP is $my = 2x$. Substituting $m = 2x/y$ in (1), we get as locus of intersection the cubic $8x^3 + 4ax^2 - ay^2 = 0$. Equation of PR is

$$y - 2mx - 2am + 2am^3 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

Equation of AQ is $y = -2mx$. Substituting $m = -(y/2x)$ in (2), we get the cubic $8x^3 + 4ax^2 - ay^2 = 0$. Therefore both intersections lie on the same cubic.

(4) Equation PR is $y - 2mx - 2am + 2am^3 = 0 \equiv \phi$,

$$\partial\phi/\partial m \equiv -2x - 2a + 6am^2 = 0, \quad \text{or} \quad m^2 = (x+a)/3a,$$

$$y - 2m(x+a) + 2am^3 = y - 2m\left\{(x+a) - am^2\right\} = y - 2m\left\{x + a - \tfrac{1}{3}(x+a)\right\}$$
$$= y - \tfrac{1}{3}\left\{4m(x+a)\right\}.$$

Envelope is $y^2 = \frac{16}{9}m^2(x+a)^2 = \frac{16}{9}.\frac{1}{3}a(x+a)(x+a)^2$, or $27ay^2 = 16(x+a)^3$.

(5) Equation AR reduces to $y(1-m^2) = 2mx$. Equation of PQ is

$$(m^2-1)\,y = 2mx - 2am.$$

These intersect on $x = \frac{1}{4}a$, a line parallel to the directrix.

6409. (Professor MATZ, M.A.)—A point is taken at random on each side of a regular hexagon, and these points are joined by straight lines to the centre of the hexagon. On each one of these lines a point is taken at random. If the *first* set of random points be joined in their respective order, find the mean area of the hexagon thus formed; also do the same if the *second* set of random points be joined in their respective order.

Solution by R. CHARTRES *and* H. W. CURJEL, M.A.

Let L, M, N, ... be the random points; then the mean value of the triangle LAM

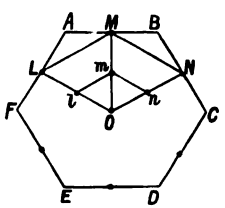
$$= \tfrac{1}{4}\text{FAB} = \tfrac{1}{4}\text{FOA}.$$

Therefore mean value of the hexagon LMN...

$$= \tfrac{3}{4} \text{ of hexagon ABC...},$$

and therefore mean value of the hexagon *lmn*...

$$= \tfrac{1}{4} \text{ of } \tfrac{3}{4} \text{ of hexagon ABC...}$$
$$= \tfrac{3}{16} \text{ of ABC....}$$



14615. (W. R. HUGHES.)—The four points determined by the intersection of the tangents and common chords of the circles of curvature at two points P, Q on an ellipse lie on a circle coaxal with the circles of curvature.

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Solution by the PROPOSER; *and* Professor SANJÁNA.

Let tangent at Q and chord at P meet in S.

Draw tangent SM to circle of curvature at P.

Remembering that the diameters parallel to TP, PR are equal (and to TQ, QR′),

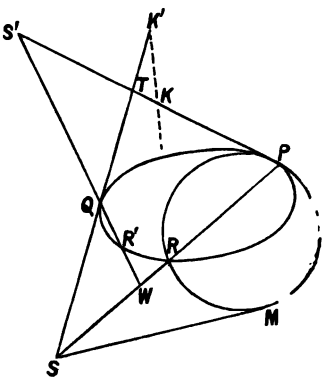
$$\frac{SM^2}{SQ^2} = \frac{SR \cdot SP}{SQ^2}$$

= ratio of squares of diameters parallel to PR, TQ

= ratio of squares of diameters parallel to TP, TQ

$$= \frac{TP^2}{TQ^2};$$

therefore SM : SQ = TP : TQ.



Also ratio of squares of tangents from W to the circles

$$= \frac{WR \cdot WP}{WR' \cdot WQ} = \text{ratio of squares of diameters parallel to PR, QR′}$$

$$= \text{,, ,, ,, TP, TQ}$$

$$= TP^2 : TQ^2,$$

and $$\frac{S'P^2}{S'Q \cdot S'R'} = \frac{TP^2}{TQ^2}$$ similarly.

Hence S, W, T, S′ are points such that the ratio of the tangents from them to the two circles are the same. Therefore they lie on a circle coaxal with the two circles.

Hence to find the radical axis of the circles of curvature, we have only to take points K, K′ on TP, TQ, such that $KP^2 = KT \cdot KS'$ and $K'Q^2 = K'T \cdot K'S$, and join them.

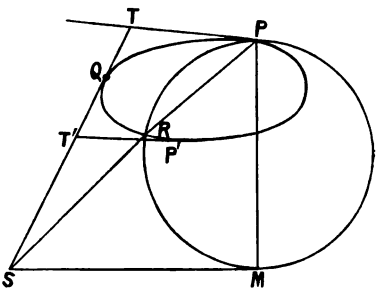
Notice also the following:— If PM be a diameter of the circle and QM meet ellipse in P′, then PP′ is a diameter of the ellipse.

For let the tangent parallel to TP meet TQ in T′. Then

$$T'P' : T'Q = TP : TQ$$

$$= SM : SQ;$$

therefore, if T′P′, SM are parallel to each other and to TP, QP′M is a straight line.



If T be on the director circle, SS′ becomes a diameter, and we have the theorem: "If the tangents at two points on an ellipse are at right angles, the line joining their centres of curvature

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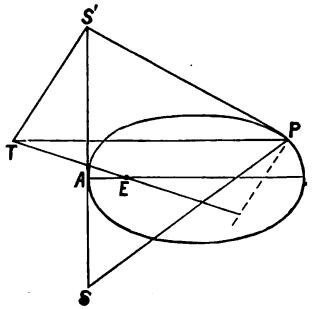
passes through the middle point of a diagonal of the quadrilateral formed by the two tangents and two chords of intersection."

When Q coincides with A, T and W move up to S′ and S, and the circle touches PS, PS′ at SS′.

Hence the theorem: "The radical axis of the circles of curvature at P and A bisects PS′ where S′ is the point of intersection of tangents at P and A.

Also the following:—"Let the parallel to the axis through P meet the line through S′ perpendicular to PS′ in T, and take E on axis, so that AE = ½ latus rectum. Then TE will meet the normal at P in the centre of curvature." This is obvious, since T, E are the centres of two circles of the coaxal system.



[Professor Jan de Vries solves the Question as follows:—
The circle of curvature at the point (x_1, y_1) is represented by

$$C_1 \equiv (b^4x_1^2 + a^4y_1^2)\,E + (a^2-b^2)\,T_1U_1 = 0, \quad E \equiv b^2x^2 + a^2y^2 - a^2b^2,$$

$$T_1 \equiv b^2x_1x + a^2y_1y - a^2b^2, \quad U_1 \equiv b^2x_1x - a^2y_1y - b^2x_1^2 + a^2y_1^2;$$

$U_1 = 0$ being the equation of the chord of curvature.

Similarly, the circle of curvature at (x_2, y_2) has the equation

$$C_2 \equiv (b^4x_2^2 + a^4y_2^2)\,E + (a^2-b^2)\,T_2U_2 = 0.$$

Evidently the pencil of circles determined by $C_1 + \lambda C_2 = 0$ contains the circle $(b^4x_2^2 + a^4y_2^2)\,T_1U_1 - (b^4x_1^2 + a^4y_1^2)\,T_2U_2 = 0$,
which passes by the intersections of T_1 and U_1 with T_2 and U_2.]

14566. (R. J. Dallas, M.A.)—If we have a set of $n-1$ equations

$$a\,(1 + x_r^2x_{r+1}^2) - b\,(x_r^2 + x_{r+1}^2) + cx_rx_{r+1} = 0 \quad (r = 1, 2, \ldots, n-1),$$

these cannot coexist with the equation $a\,(1 + x_n^2x_1^2) - b\,(x_n^2 + x_1^2) + cx_nx_1 = 0$ unless a certain relation holds between a, b, c when the equations are indeterminate. Find these relations for the cases $n = 3, 4, 5, 6$.

Solution by the Proposer.

When there are three equations in x_1, x_2, x_3 we may proceed as follows:—
Let x_1, x_2, x_3 be roots of $x^3 + px^2 + qx + r = 0$. The one equation may be written $a\,(1 + r^2/x^2) - b\,(p^2 - 2q - x^2) - cr/x = 0$,
or $bx^4 + x^2\left[a - bp^2 + 2bq\right] - crx + ar^2 = 0$.
This must be identical with $b\,(x-p)(x^3 + px^2 + qx + r) = 0$.
Comparing coefficients

$$a - bp^2 + 2bq = bq - bp^2, \quad -cr = br - bpq, \quad ar = -bp,$$

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we find we can eliminate p, q, r from these equations, and obtain

$$a^2 = b(b+c).$$

For $n > 3$ this method will not serve. We resort to geometry. Let $x^2+y^2 = R^2$, $(x-d)^2+y^2 = r^2$ be two circles. A chord of the first is

$$x\cos\tfrac{1}{2}(\theta+\phi) + y\sin\tfrac{1}{2}(\theta+\phi) = R\cos\tfrac{1}{2}(\theta-\phi).$$

If this touches the second circle, we have

$$R\cos\tfrac{1}{2}(\theta-\phi) - r = d\cos\tfrac{1}{2}(\theta+\phi),$$

whence $$(R-d)\cos\tfrac{1}{2}\theta\cos\tfrac{1}{2}\phi + (R+d)\sin\tfrac{1}{2}\theta\sin\tfrac{1}{2}\phi = r.$$

If we write x_1 for $\tan\frac{1}{4}\theta$, x_2 for $\tan\frac{1}{4}\phi$, we get

$$(R-d)(1-x_1^2)(1-x_2^2) + 4(R+d)x_1x_2 = r(1+x_1^2)(1+x_2^2),$$

or $$(R-d-r)(1+x_1^2x_2^2) - (R-d+r)(x_1^2+x_2^2) + 4(R+d)x_1x_2 = 0,$$

which is an equation of the form given.

Now, if θ, ϕ, ψ be the angular coordinates of the vertices of a polygon inscribed in the first circle, and circumscribed to the second, we know (see, for instance, Casey's *Sequel to Euclid*, section on "Coaxal Circles") that a certain relation must exist between R, r, and d, and that then an infinity of such polygons exist. Hence the necessary relation between a, b, and c can be deduced by taking one vertex on the line of centres; that is, $x_1 = 0$.

For four letters, the third vertex of quadrilateral must also be on line of centres, so that $x_3 = \tan\frac{1}{4}\pi = 1$. Putting $x_1 = 0$, we find $x_2 = \sqrt{(a/b)}$. Substitute this in $a(1+x_2^2x_3^2) - b(x_2^2+x_3^2) + cx_2x_3 = 0$, and put $x_3 = 1$.

Hence $$a(1+a/b) - b(1+a/b) = -c\sqrt{(a/b)}, \quad \text{or} \quad (a^2-b^2)^2 = c^2ab.$$

For a pentagon, we find $$x_3 = -\frac{c\sqrt{ab}}{a^2-b^2}.$$

If we substitute this in $a(1+x_3^2x_4^2) - b(x_3^2+x_4^2) + cx_3x_4 = 0$,

the equation will give us as roots x_2 or x_4, which is $-x_3$. This equation may be written

$$x_4^2\left\{\frac{c^2a^2b}{(a^2-b^2)^2} - b\right\} - \frac{c^2\sqrt{ab}}{a^2-b^2}x_4 + a - \frac{b^2c^2a}{(a^2-b^2)^2} = 0.$$

The product of the roots is thus $\sqrt{(a/b)} + (c\sqrt{ab})/(a^2-b^2)$ or $ac/(a^2-b^2)$.

Hence $$a/b\,\frac{(a^2-b^2)^2 - b^2c^2}{a^2c^2-(a^2-b^2)^2} = \frac{ac}{a^2-b^2}, \quad \text{or} \quad \frac{(a^2-b^2)^2-b^2c^2}{a^2c^2-(a^2-b^2)^2} = \frac{bc}{a^2-b^2}.$$

For a hexagon we must have $x_4 = 1$, as with the quadrilateral $x_3 = 1$. Substituting $(-c\sqrt{ab})/(a^2-b^2)$ for x_3, 1 for x_4, in equation

$$a(1+x_3^2x_4^2) - b(x_3^2+x_4^2) + cx_3x_4 = 0,$$

we get $$(a-b)\left\{1 + \frac{c^2ab}{(a^2-b^2)^2}\right\} = \frac{c^2\sqrt{ab}}{a^2-b^2}.$$

Or, squaring, $$\left\{(a^2-b^2)^2 + c^2ab\right\}^2 = c^4ab(a+b)^2.$$

[Mr. R. F. Davis remarks:—The above are particular forms of the

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standard porismatic equations given in WOLSTENHOLME; and the relations between the constants are as follows :—

$$n = 3, \qquad a^2 - b(b+c) = 0,$$
$$n = 4, \qquad c(a^2 - b^2) = 0,$$
$$n = 5, \qquad \{a^2 - b(b+c)\}^3 - 4bc(a^2 - b^2)\{a^2 - (b^2 + bc + \tfrac{1}{4}c^2)\} = 0.$$

They present themselves in the consideration of the conditions under which it is possible to circumscribe a triangle or other figure to the parabola $y^2 = 4px$, whose vertices lie on the conic

$$ax^2 - by^2 + px(2b+c) + ap^2 = 0.]$$

14724. (R. F. MUIRHEAD.)—Through a given point E draw a line QEP meeting AB produced in Q and BC produced in P so that BQ = CP, A, B, C being given points.

I. *Solution by* A. M. NESBITT, M.A.

Draw EG parallel to AB. Take GH = GE, and find P such that

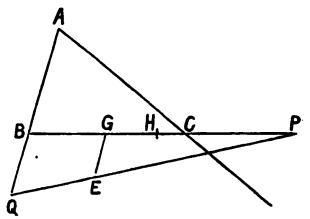
$$HP.PC = BC.EG.*$$

Join PE and produce to Q.

Then EG/PC = PH/BC = PG/PB;

therefore EG/GP = CP/PB;

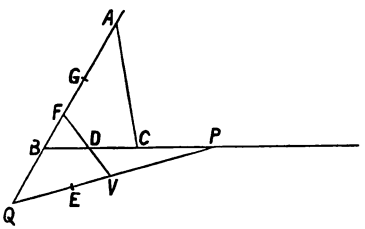
therefore CP = BQ.



II. *Solution by the* PROPOSER.

It is a well-known theorem that, if lines be drawn like PQ, making CP = BQ, they envelop a parabola. And it is clear that the parabola would touch BC at C, and BA in G, a point such that BG = BC. Thus the line joining D and F, the mid-points of BC and BG, would be the tangent to the vertex, and the focus would be the circumcentre of BCG.

The problem therefore reduces to that of drawing a tangent



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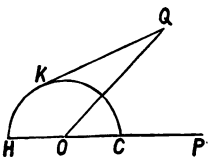
* This is easily done by describing a semicircle on HC, drawing any tangent KQ such that

$$KQ^2 = BC.EG,$$

and taking OP = OQ, so that

$$PH.PC = QK^2 = BC.EG.$$



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through a given point E to a parabola of which the focus and vertex are given. This is solved in books on geometrical conics. (See DREW, *Parabola*, Prop. XI.) There are two solutions.

Note.—If V be the mid-point of PQ, then, by the converse of MENELAUS' theorem, F, D, V are collinear. Thus the problem of the Question is equivalent to this: Through a given point E, to draw a line PVQ which shall meet two given lines AB, BC in Q and P, and be bisected by a third given line FD, which is parallel to the bisector of the angle made by the other two.

14574. (Professor A. DROZ-FARNY.)—La bissectrice intérieure de l'angle A du triangle ABC coupe en D et E le côté BC et la circonférence circonscrite au triangle. Soit I le centre du cercle inscrit. Calculer le rapport AI/DE. Que devient ce rapport lorsque les côtés du triangle sont en progression arithmétique, BC étant le côté moyen ?

Solution by Professor K. J. SANJÁNA; *and* W. J. GREENSTREET, M.A.

We have $AI = r \operatorname{cosec} \frac{1}{2}A$

$$= r(bc)^{\frac{1}{2}} \div \left[(s-b)(s-c)\right]^{\frac{1}{2}};$$

also $$\frac{DE}{BD} = \frac{\sin \frac{1}{2}A}{\sin C}$$

gives

$$DE = \frac{ca}{b+c} \cdot \frac{\sin \frac{1}{2}A}{\sin C}$$

$$= a^2 (bc)^{\frac{1}{2}} \div 2(b+c)\left[s(s-a)\right]^{\frac{1}{2}}.$$

Hence the ratio

$$AI/DE = 2(b+c)(s-a)/a^2.$$

If the sides are in A.P., a being the mean,

$$b+c = 2a \quad \text{and} \quad 2(s-a) = b+c-a = a;$$

hence

$$AI/DE = 2.$$

14558. (Rev. J. CULLEN.)—Show that the method of Quest. 14506 can be used to determine the prime or composite character of all numbers of the form $4n+1$ of eleven figures (or under). Apply the method to express $3 \cdot 2^{30}+1$ as the sum of two squares and to determine if this partition is unique.

Solution by the PROPOSER.

If N is expressible in the form ax^2+by^2, we can solve for x (or y) to mod p, where p stands successively for 2^t, 3^s, 5, 7, 11, &c., and by combining certain congruences we obtain the α's and β's, and the work pro-

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ceeds as in Quest. 14506. Further, if the determinant of the form $(a, 0, b)$ be an "idoneal" number (*cf.* MATHEWS, Art. 213, where sixty-five such numbers are given), we shall factorize N or determine it as prime on proceeding to the required limit.

Now, all *primes* of the form $4n+1$ are expressible in the form a^2+b^2, and, taking a odd, we have 4, 2 or 3, 8 or 9, cases in 32, 5, 17. Thus, with 8 or 9 symbols and 8 or 12 arrangements, we obtain the modulus 2,720, so that for a number about the size of 10^{11} we should proceed only to 113 divisions.

Taking

$$N = 3.2^{30}+1 = 3,221,225,473 \equiv 1 \pmod{64} \equiv 3 \pmod 5 \equiv 6 \pmod{17};$$

therefore $a \equiv \pm 1, 7 \pmod{32} \equiv \pm 2 \pmod 5 \equiv \pm 2, 5, 6, 7 \pmod{17}$.

$$P = 5.32 = 160, \quad Q = 17, \quad \alpha_1 = 7.$$

Hence $160\beta_k^{(q)} + \alpha_q \equiv \beta_k \pmod p$, $160\beta_k^{i} + \alpha_1 \equiv r_k \pmod p$,

and $$160\beta_1^{(q)} + \alpha_q \equiv \rho_q \pmod{17}.$$

Now, observing that $a \equiv a_1, \ldots, a_8 \pmod{160} \equiv \beta_1, \ldots, \beta_8 \pmod{17}$, we have

β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	α_2	α_3	α_4	α_5	α_6	α_7	α_8
2	5	6	7	10	11	12	15	33	57	63	97	103	127	153
β_1^{i}	β_2^{i}	β_3^{i}	β_4^{i}	β_5^{i}	β_6^{i}	β_7^{i}	β_8^{i}	β_1^{ii}	β_1^{iii}	β_1^{iv}	β_1^{v}	β_1^{vi}	β_1^{vii}	β_1^{viii}
9	7	12	0	15	3	8	6	15	14	1	1	5	4	10

We can now draw up tables for $[r_k]_{k=8}^{k=1}$ and $[\rho_q]_{q=8}^{q=1}$ for the different primes to be employed, and as an example the work for the prime 13 is given. $N \equiv 11 \pmod{13}$; therefore $a \equiv \pm 1, 5, 6 \pmod{13}$, $PQ \equiv 3 \pmod{13}$,

$$\left.\begin{matrix} r_1, r_2, r_3, r_4, r_5, r_6, r_7, r_8, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_7, \rho_8 \\ 4, 9, 3, 7, 2, 6, 0, 5, 2, 9, 2, 10, 6, 0, 11 \end{matrix}\right\} \pmod{13}.$$

Whence, as described in Quest. 14506, we have

0	3	6	9	12	2	5	8	11	1	4	7	10	0
^	123		23	2	1 3	c23	1	23	1	1 3		12	^
	5	e56	g	456	f 6	45	6	4	4 6	5	4h	5	4a6
	8	7	8	7	8	7	7	bd	78		78		8
v													v

$a, b, \ldots, h$ being used to give the initial divisions in each arrangement corresponding to $\rho_1, \rho_2, \ldots, \rho_8$. Now

$$a = 2720.x + 160\beta_k^{(q)} + \alpha_q, \quad (\beta_k^{(q)} = \beta_1^{(q)} + \beta_k^{i} - \beta_1^{i}),$$

and we need only proceed to to the twenty-second division, since $\sqrt{N} \div 2720 < 22$. Supposing $x_k^{(q)}$ to mean that the symbol k appeared in all the strips in the qth arrangement at the xth division, the following results were obtained :—

$$17_5^{i},\ 6_8^{ii},\ 17_5^{ii},\ 1_1^{iv},\ 10_1^{v},\ 3_3^{vi},\ 6_4^{vi},\ 11_5^{vii},\ 16_4^{vii}$$

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(thus, for instance, 17^{11}_{5} gives $a = 17.2720 + 160(15 + 15 - 9) + 33 = 49{,}633$), giving the following numbers as going through all mods up to 37 :—

Mod	N	48,647	18,033	49,633	2,943	27,457	9,543	15,783	31,647	42,847
41	39	21	34	23	32	**28**	31	**39**	**36**	**2**
43	13	**14**	16	11	**19**		40			
47	9		**32**	1			**2**			

The numbers in black type fail to conform with $N - a^2 = \rho \pmod{p}$, where ρ is a 2-ic residue of p. Thus the partition $N = 27528^2 + 49633^2$ is unique, and N is therefore prime.

14604. (Professor Umes Chandra Ghosh.) — If a conic circumscribing a quadrilateral ABCD be inscribed in a quadrilateral *abcd*, *ab* touching the conic at D, *bc* at A, *cd* at B, and *da* at C, prove that the following quartets of lines are concurrent and harmonically conjugate :— (1) DB, AC, *bd*, *ac* (see Salmon's *Conics*, p. 242); (2) DA, BC, *ac*, *ef*; (3) AB, DC, *bd*, *ef*; and that the three diagonals of either quadrilateral form a self-conjugate triangle with respect to the conic.

Solution by W. J. Greenstreet, M.A.

For these familiar properties see Cremona's *Projective Geometry*, § 170, p. 141, or for the circle see Russell's *Pure Geometry*, §§ 16 and 18, p. 33.

14489. (Professor N. Bhattacharyya.)—A triangle ABC is such that its circumcentre is coincident with the centre of its Brocard ellipse. The lines AΩ, BΩ, CΩ meet the opposite sides in D, E, F, respectively; and likewise the lines AΩ′, BΩ′, CΩ′ meet the opposite sides in D′, E′, F′, respectively. Let the lines EF, E′F′; FD, F′D′; DE, D′E′ meet, produced, if necessary, in P, Q, R, respectively. Prove that (1) the lines AP, BQ, CR are concurrent; (2) the sum of the squares of the axes of the ellipse $= 4(5R^2 - 2\Delta \cot \varpi)$, where R denotes the circumradius, Δ the area, and ϖ the Brocard angle of the triangle.

Solution by W. J. Greenstreet, M.A.

The circumcentre is $R\cos A, \ldots, \ldots$. The Brocard centre is $R\sin^2\omega\,(b^2 + c^2)/bc, \ldots, \ldots$; therefore

$$\sin^2\omega = bc\cos A/(b^2 + c^2) = (b^2 + c^2 - a^2)/[2(b^2 + c^2)],$$

and by symmetry = two similar expressions $= \Sigma a^2/4\Sigma a^2 = \frac{1}{4}$; therefore $\omega = 30°$, and the triangle is equilateral.

For $b^2 + c^2 = 4bc\cos A - 2b^2 + 2c^2 - 2a^2$; therefore $b^2 + c^2 = 2a^2$, and two similar expressions; whence $a = b = c$. Therefore the sum of squares of the axes $= 4(a_1^2 + b_1^2) = 4(4R^2\sin^4\omega + R^2\sin^2\omega) = 4(\frac{1}{4}R^2 + \frac{1}{4}R^2) = 2R^2$. Also $2\Delta\cot\omega = 2 \cdot \frac{1}{2}a^2 \cdot \frac{1}{2}\sqrt{3} \cdot \sqrt{3} = \frac{3}{2}a^2 = \frac{9}{2}R^2$ (as $a = 2R\cdot\frac{1}{2}\sqrt{3} = R\sqrt{3}$); therefore $4(5R^2 - \frac{9}{2}R) = 2R^2$; therefore (2) is proved.

(1) is true for *any* two points P and Q (Smith's *Conics*, xii., p. 310).

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6835. (Col. A. R. Clarke, C.B., F.R.S.)—Two equal and similar arcs of a circle, of angular extent α, are drawn on a piece of paper. They intersect: what are the respective chances of one and two intersections?

Solution by H. M. Taylor, M.A., F.R.S.

The following solution is based on the hypothesis that all angles of intersection of the two arcs are equally probable and that all possible positions of each arc in its own circle are equally probable.

Let θ be the angle at which the two circles of which the two arcs form parts intersect.

First, let the given arcs be less than semicircles or $\alpha < \pi$. Then, if α be greater than θ, the middle point of either of the given arcs must lie in an arc of angle $\alpha - \theta$ in order that the given arc may pass through both intersections of the two circles.

Therefore the chance of the arcs intersecting twice

$$= \int_0^\alpha \left(\frac{\alpha-\theta}{\alpha}\right)^2 \frac{d\theta}{\pi} = \frac{\alpha}{3\pi}.$$

Secondly, let the given arcs be greater than semicircles or $\alpha > \pi$. Then, if θ be less than $2\pi - \alpha$, the middle point of each of the given arcs must lie in an arc of angle $\alpha - \theta$, and, if θ be greater than $2\pi - \alpha$, the middle point of each of the given arcs must lie in one or other of two arcs of angles $\alpha - \theta$ and $\alpha - (2\pi - \theta)$ respectively, in order that the given arc may pass through both intersections of the two circles.

Therefore the chance of the arcs intersecting

$$= \int_0^{2\pi-\alpha} \left(\frac{\alpha-\theta}{\alpha}\right)^2 \frac{d\theta}{\pi} + \int_{2\pi-\alpha}^{\pi} \left(\frac{2\alpha-2\pi}{\alpha}\right)^2 \frac{d\theta}{\pi} = \frac{\alpha}{3\pi} + \frac{4}{3}\frac{(\alpha-\pi)^3}{\alpha^2\pi}.$$

[For another solution of this Question, see pp. 92, 93 of the present volume.]

14621. (H. W. Curjel, M.A.)—If quadrilaterals can be at the same time inscribed in a conic C_1 and circumscribed to a conic C_2, prove that triangles can be at the same time inscribed in C_1 and circumscribed to the conic C_3, which is the polar reciprocal of C_1 with respect to C_2; and, if P is the common vertex of such a triangle and quadrilateral, the vertex of the quadrilateral opposite to P and the side of the triangle opposite to P are pole and polar with respect to C_2.

Solution by R. J. Dallas.

The conics C_1 and C_2 can, with perfect generality, be taken to be

$$b^2x^2 + a^2y^2 = a^2b^2 \ (C_1), \qquad x^2 + y^2 = c^2 \ (C_2),$$

and C_3 will be thus $\qquad a^2x^2 + b^2y^2 = c^4.$

Any tangent to C_2 is $\qquad x\cos\theta + y\sin\theta = c.$

Taking one vertex of quadrilateral inscribed to C_1 at $(a, 0)$, we readily find (or it is geometrically evident) that condition for this quadrilateral touching C_2 is $c^2(a^2 + b^2) = a^2b^2$; and of course, this condition being fulfilled, wherever the vertex of the quadrilateral is taken on C_1, it will close

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up. So any tangent to C_2 is $ax\cos\theta + by\sin\theta = c^2$. Making this pass through $(a, 0)$, we find $\cos\theta = c^2/a^2$. Substituting this value, we find the tangent cuts C_1 again at a point whose x coordinate is

$$a(c^4a^4 - a^4b^4 + b^4c^4) + (c^4a^4 + a^4b^4 - b^4c^4).$$

Now, if the (isosceles) inscribed triangle thus obtained touches C_2, this x must $= -c^2/a$. Hence

$$a^2(c^4a^4 - a^4b^4 + b^4c^4) = -c^2(c^4a^4 + a^4b^4 - b^4c^4).$$

Removing an irrelevant factor $a^2 + c^2$, we find $c^4a^4 = b^4(a^2 - c^2)^2$, whence $c^2 = a^2b^2/(a^2 + b^2)$, the same condition as for quadrilateral porism between C_1 and C_2.

This proves (1); for (2) let θ and ϕ be two points on C_1; the chord joining them is $x/a\cos\frac{1}{2}(\theta+\phi) + y/b\sin\frac{1}{2}(\theta+\phi) = \cos\frac{1}{2}(\theta-\phi)$.

This will touch C_2, which is $a^2x^2 + b^2y^2 = a^4b^4/(a^2+b^2)^2$,

if $\{b/a\cos\frac{1}{2}(\phi+\theta)\}^2 + \{a/b\sin\frac{1}{2}(\theta+\phi)\}^2 = \{(a^2+b^2)/ab\cos\frac{1}{2}(\phi-\theta)\}^2$,

which reduces to $\cos\theta\cos\phi(a^2/b^2+1) + \sin\theta\sin\phi(b^2/a^2+1) + 1 = 0$.

Given θ, this equation gives two (geometrically different) values of ϕ, viz. ϕ_1 and ϕ_2; and we easily prove that

$$\cos\tfrac{1}{2}(\phi_1+\phi_2) : \sin\tfrac{1}{2}(\phi_1+\phi_2) : \cos\tfrac{1}{2}(\phi_1-\phi_2)$$
$$= \cos\theta(a^2/b+1) : \sin\theta(b^2/a^2+1) : -1.$$

Hence the side of the triangle whose vertex is θ has the equation (regarded as a chord of C_1)

$$x/a\cos\theta(a^2/b^2+1) + y/b\sin\theta(b^2/a^2+1) = -1$$

or
$$x(-a\cos\theta) + y(-b\sin\theta) = a^2b^2/(a^2+b^2),$$

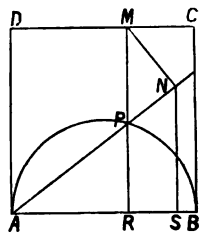
and is thus the polar, with respect to C_2, of the point $(\theta+\pi)$, which is the opposite vertex of the quadrilateral inscribed in C_1.

14672. (Salutation.) — Within the square ABCD the semicircle AIB is described; from A a straight line is drawn, cutting the semicircle in P; through P another line is drawn, perpendicular to AB, cutting CD in M; and finally MN is drawn perpendicular to AP produced. Find the locus of N.

Solution by J. H. Taylor, M.A.; Rev. T. Wiggins, B.A.; *and many others.*

Draw NS perpendicular on AB. Let $AB = a$.

Let $\angle PAB = \theta$.



Then $AP = a\cos\theta$, $AR = a\cos^2\theta$,

$PR = a\sin\theta\cos\theta$, $MP = a(1-\sin\theta\cos\theta)$,

$PN = a(1-\sin\theta\cos\theta)\sin\theta$,

$AN = AP + PN$,

or
$$\rho = a(\cos^2\theta + \sin\theta),$$

the polar equation with A origin and AB the initial line.

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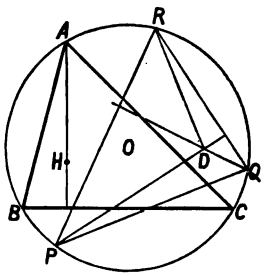
14613 & 14798. (14613.) (Rev. J. Cullen.)—Two points P and Q lie on the circumcircle of a given triangle ABC, and subtend a constant angle at O, the centre. Prove that the locus of the intersection of the Simson-lines of P and Q is a hypotrochoid.

(14798.) (Professor Umes Chandra Ghosh.)—Find the envelope of Simson's lines of a given triangle with respect to points on its circumference.

Solution by C. E. Youngman, M.A.

Draw PD, QD parallel to the pedal lines of P, Q respectively. Then the locus of D will be the same kind of curve as the required locus; for, if H be the orthocentre of ABC, the pedals bisect HP and HQ, and therefore meet at the mid-point of HD. We know that the angle PDQ is equal to that subtended by PQ (on the other side) at the circumference ABC; hence PR and QR, drawn perpendicular to QD and PD respectively, will meet on that circumference. Now D is the orthocentre of PQR; therefore RD = twice the distance of O from PQ, and is constant. Also, by symmetry, RD is parallel to the pedal line of R; hence the angle between RD and AH (which is the pedal of A) = APR = $\frac{1}{2}$AOR. Thus RD turns round R half as fast as OR turns round O, and in the opposite direction, as the figure shows; therefore D describes a retrograde epicyclic—the same thing as a hypotrochoid, for which the fixed circle (centre O) has a radius (ρ) three times that of the rolling circle. And ρ may be either $\frac{3}{2}$OR or $\frac{3}{4}$RD, the corresponding distance (d) of tracing point from rolling centre being RD or OR.



As a limiting case, make the constant angle zero; then P, Q coincide, RD = 2OP, ρ = 3OP, d = OP; and the curve—a tricuspid hypocycloid—touches PD. [See *Reprint*, Vol. III., p. 97; also Vol. XXXIV., for another solution of Quest. 14798.]

[Speaking in regard to Quest. 14798, Mr. Tucker observes:—This envelope (a tricuspidal quartic) has been discussed by Steiner; *cf.* results given in Salmon's *Higher Plane Curves*, p. 271 (ed. 1873).

Or, taking the equation to a Wallace-line, as given in Notes in *Edinburgh Math. Soc. Proc.*, Vols. XIV., XV., and dividing all through by $\cos\theta$, we have ($t = \tan\theta$)

$$t^3\, a\alpha \sin B \sin C + t^2 [a\alpha \sin(B-C) + b\beta \sin B \cos C - c\gamma \sin C \cos B] - t [a\alpha \cos B \cos C + b\beta \cos(B-C) + c\gamma \cos(B-C)] + b\beta \sin C \cos B - c\gamma \sin B \cos C = 0.$$

This is of the form $Pt^3 + Qt^2 + Rt + S = 0$,

and for envelope (also) $3Pt^2 + 2Qt + R = 0$.

Hence envelope is

$$12P(R^2 - 3QS)^2 - 4Q(R^2 - 3QS)(QR - 9PS) + R(QR - 9PS)^2 = 0$$

or $$4PR^3 - 18PQRS - Q^2R^2 + 4Q^3S + 27P^2S^2 = 0.$$

This is only noteworthy in that the envelope is given in trilinears.]

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14657. (Professor LANGHORNE ORCHARD, M.A., B.Sc.)—Show that

$$3\{1^5+2^5+3^5+\ldots+n^5-(1+2+3+\ldots+n)^3\}$$
$$=(1+2+3+\ldots+n)^3-(1+2+3+\ldots+n)^2,$$

n being a positive integer.

Solution by H. J. WOODALL, A.R.C.S.; Rev. T. WIGGINS, B.A.; *and many others.*

This may be written $3(S^5-S_1^3)=S_1^3-S_1^2$,

and $$S^5=\tfrac{1}{3}n^3(n+1)^3-\tfrac{1}{12}n^2(n+1)^2=\tfrac{4}{3}S_1^3-\tfrac{1}{3}S_1^2;$$

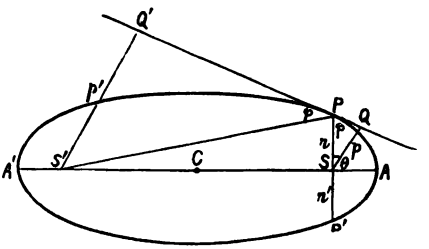
whence the theorem.

14602. (Professor N. BHATTACHARYYA.)—A light string *just fits* round an ellipse (eccentricity $= e$), and is under the action of a central force towards a focus, varying inversely as the distance. If T and T′ be the tensions at the extremities of any focal chord, show that $\epsilon^{-(T/\mu)}+\epsilon^{-(T'/\mu)}$ is constant. If N be the normal pressure on the ellipse at any point whose distance from the centre of force is r, prove that

$$N=\frac{b\mu}{\sqrt{(2ar-r^2)}}\left\{\frac{1}{r}+\frac{a}{(2ar-r^2)}\log\frac{r}{a(1-e)}\right\}.$$

Solution by CONSTANCE I. MARKS, B.A.

Let the ellipse be referred to S as pole. An element ds of the string at P is in equilibrium under the tensions T and $T+dT$ at the extremities, the central force $f=-\mu/r$ say, and the reaction of the curve $-N$ acting along the radius of curvature at P since the curve is assumed smooth.



With usual notation, resolving along the tangent at P,

$$dT/ds=\mu/r\cos\phi \quad \text{or} \quad dT/dr\,dr/ds=\mu/r\,dr/ds;$$

$\therefore\ dT=\mu\,dr/r$, $T=\mu\log r-\mu\log C$ where C is constant $=-\mu\log C/r$;

therefore $$\epsilon^{-(T/\mu)}=C/r.$$

Similarly, if T′ be the tension at P′, the other end of the focal chord through P, $\epsilon^{-(T'/\mu)}=C/r'$. Hence

$$\epsilon^{-(T/\mu)}+\epsilon^{-(T'/\mu)}=C(1/r+1/r')=2C/l \text{ from polar equation to curve}$$
$$=\text{constant}.$$

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Let T_0 be the tension at A. Then

$$\frac{\epsilon^{-(T/\mu)}}{\epsilon^{-(T_0/\mu)}} = \frac{C/r}{C/[a(1-e)]}; \quad \text{therefore} \quad \epsilon^{-(T/\mu)} = [a(1-e)]/r\ \epsilon^{-(T_0/\mu)};$$

therefore $$T = \mu \log r/[a(1-e)] + T_0.$$

Now A′ and A are easily seen to be respectively the points of maximum and minimum tension, and, assuming that for a string that "just fits" the minimum tension is zero, we have T_0 (the tension at A) $= 0$; therefore

$$T = \mu \log r/[a(1-e)] \quad \text{.....(i.).}$$

By normal resolution $$T/\rho + \mu/r \sin\phi = N,$$

that is $$T/\rho + \mu p/r^2 = N \quad \text{.....(ii.).}$$

From the similar triangles PQS, PQ′S′, and the known formula $\rho = r\,dr/dp$, we obtain

$$p/r = \frac{b}{\sqrt{(2ar-r^2)}}, \quad \rho = \frac{(2ar-r^2)^{\frac{3}{2}}}{ab}.$$

Substituting in (ii.) for $T\rho$ and p/r,

$$N = \frac{\mu \log r/[a(1-e)]}{[(2ar-r^2)^{\frac{3}{2}}]/ab} + \frac{\mu b}{r\sqrt{(2ar-r^2)}}$$

$$= \frac{\mu b}{\sqrt{(2ar-r^2)}} \left\{ \frac{1}{r} + \frac{a}{(2ar-r^2)} \log \frac{r}{a(1-e)} \right\}.$$

14662. (R. C. Archibald, M.A.)—In the xy-plane, passing through the origin S, and with its centre on the x-axis, is a circle of radius a. On this circle is rolled an equal circle whose plane is always perpendicular to the xy-plane. The point P of this rolling circle, which was at the beginning of the motion in coincidence with S, traces out a curve C. Show that (1) the curve C lies both on a sphere of radius $a\sqrt{2}$ and on the cone $x^2 + y^2 = z^2$; (2) the orthogonal projection of C on the xy-plane is the cardioid whose axial circle is the given fixed circle; (3) the projection of C on the yz-plane is the cuspidal quartic called by the French a *toupie* or *piriforme*; (4) the projection of C on the xz-plane is a parabola.

I. *Solution by* Rev. J. Cullen.

Let θ be the angle subtended by S, and the point of contact of the two circles at the centre of the fixed circle; then it is easy to see that

$$x = -a\cos\theta(1-\cos\theta), \quad y = a\sin\theta(1-\cos\theta) \quad \text{.....(i.), (ii.),}$$

$$z = a(1-\cos\theta) \quad \text{.....(iii.).}$$

(1) $$(x-a)^2 + y^2 + (z-a)^2 = 2a^2, \quad \text{also} \quad x^2 + y^2 = z^2;$$

hence C is the intersection of these surfaces, the radius of the sphere being $a\sqrt{2}$.

(2) From (i.) and (ii.), we obtain the cardioid

$$r = a(1-\cos\theta).$$

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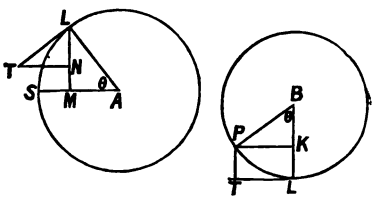
(3) The elimination of θ between (ii.) and (iii.) gives $z^3(2a-z)-a^2y^2=0$, the origin being a cusp.

(4) Similarly we obtain $z^2-az-ax=0$, a parabola.

II. *Solution by* JAMES BLAIKIE, M.A.



Let A be the centre of the horizontal circle, B that of the vertical circle, a the length of the radius of each circle. When BP makes an angle θ with the vertical, let L be the point of contact of the circles, LT the common tangent, PT the perpendicular from P to TL, and PK the perpendicular from P to BL. In the horizontal circle draw LM perpendicular to SA and TN perpendicular to LM. Then the coordinates of P are $x = \mathrm{SM}-\mathrm{TN} = a(1-\cos\theta)-\mathrm{TL}\sin\theta$; but $\mathrm{TL} = \mathrm{PK}$;

therefore $\quad x = a(1-\cos\theta-\sin^2\theta) = a(\cos^2\theta-\cos\theta)$,

$$y = \mathrm{NM} = a\sin\theta(1-\cos\theta), \quad z = \mathrm{PT} = a(1-\cos\theta).$$

These values satisfy the equation $(x-a)^2+y^2+(z-a)^2 = 2a^2$, which is that of a sphere of radius $a\sqrt{2}$. They also satisfy the equation $x^2+y^2=z^2$.

Taking the values of x and y, we find, as the equation of the projection on the plane xy, $x^2+y^2 = a^2(1-\cos\theta)^2$ or $r = a(1-\cos\theta)$, the cardioid whose axis is a diameter of the given horizontal circle. The projection on yz is $(y/z)^2+(1-z/a)^2 = 1$ or $a^2y^2-2az^3+z^4=0$. That on zx is the parabola $z^2 = a(x+z)$.

14685. (R. TUCKER, M.A.)—Find the H.C.F. of

$$\begin{vmatrix} \cos(C-\omega)\cos(A-\omega) & -\cos\omega\cos(A-\omega) & \cos^2\omega \\ \cos^2\omega & \cos(A-\omega)\cos(B-\omega) & -\cos\omega\cos(B-\omega) \\ -\cos\omega\cos(C-\omega) & \cos^2\omega & \cos(B-\omega)\cos(C-\omega) \end{vmatrix},$$

and

$$\begin{vmatrix} c\cos(A-\omega) & -a\cos\omega\cos(A-\omega) & a\cos^2\omega \\ a & b\cos(A-\omega) & -b\cos\omega \\ b\cos(C-\omega) & c\cos^2\omega & c\cos(B-\omega)\cos(C-\omega) \end{vmatrix}.$$

Solution (1) *by the* PROPOSER; (2) *by* CONSTANCE I. MARKS, B.A.

(1) (i.) $= [\Pi\cos(A-\omega)+\cos^3\omega]^2 = (P)^2$,

(ii.) $= c[bc\cos(A-\omega)+a^2\cos\omega][P]$

$$= c\left[\frac{b^2c^2+a^2bc\cos A}{\lambda}+\frac{a^2k}{2\lambda}\right][P] = c\lambda[P];$$

therefore H.C.F. is P, which

$$= \sin^3\omega\Pi\sin A\operatorname{cosec}^4\omega = \Pi\sin A\operatorname{cosec}\omega.$$

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(2) For brevity write $\cos(A-\omega) = A'$, $\cos(B-\omega) = B'$, $\cos(C-\omega) = C'$. Thus first determinant is

$$\Delta = \begin{vmatrix} C'A' & -\cos\omega A' & \cos^2\omega \\ \cos^2\omega & A'B' & -\cos\omega B' \\ -\cos\omega C' & \cos^2\omega & B'C' \end{vmatrix}.$$

Multiply third column by A', multiply second column by $\cos\omega$, and add to new third column; therefore

$$\Delta = 1/A' \begin{vmatrix} C'A' & -\cos\omega A' & 0 \\ \cos^2\omega & A'B' & 0 \\ -\cos\omega C' & \cos^2\omega & \cos^3\omega + A'B'C' \end{vmatrix}$$

$$= (\cos^3\omega + A'B'C') \begin{vmatrix} C' & -\cos\omega \\ \cos^2\omega & A'B' \end{vmatrix} = [A'B'C' + \cos^3\omega]^2.$$

Treating second given determinant, Δ' say, in exactly same way,

$$\Delta' = c[\cos^3\omega + A'B'C'][bcA' + a^2\cos\omega];$$

therefore H.C.F. of Δ and $\Delta' = \cos(A-\omega)\cos(B-\omega)\cos(C-\omega) + \cos^3\omega$.

14667. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Factorize *completely* (into prime factors) $N = 7^{35} + 1$.

Solution by H. J. WOODALL, A.R.C.S.; *the* PROPOSER; *and* JAMES BLAIKIE, M.A.

The common algebraical factors of N and the resolution by the sept-Aurifeuillian function gives

$$N = 8;\ 11.191;\ 113;\ 911;\ 421.12128131;\ 71.603926681.$$

There still remain two factors above 9 million. Each of these was tested by division by all the primes of form $70m+1$ below 10,000. This proves the smaller number to be prime. The larger was further tested by the following method (a modification of Mr. BIDDLE's method). Writing $(70m+1)(70n+1) = 603926681$, we find $mn = 123250-k$, $m+n = 24+70k$. Then we get the following table:—

k	mn	$m+n$	$(m+n)^2$	$4mn$	$(m-n)^2$?	Test for Square.
10	123240	724	524176	492960	31216	no
11	123239	794	630436	492956	137480	no
12	123238	864	746496	492952	253544	no
13	123237	934	872356	492948	379408	no
14	123236	1004	1016064	492944	523120	no

Therefore 603926681 is prime.

Note.—k is less than 15 because a greater value would still further increase $m+n$; and $m+n$ must be $\not> 1005$, because, as is proved already, there is no factor $< 10{,}000$.

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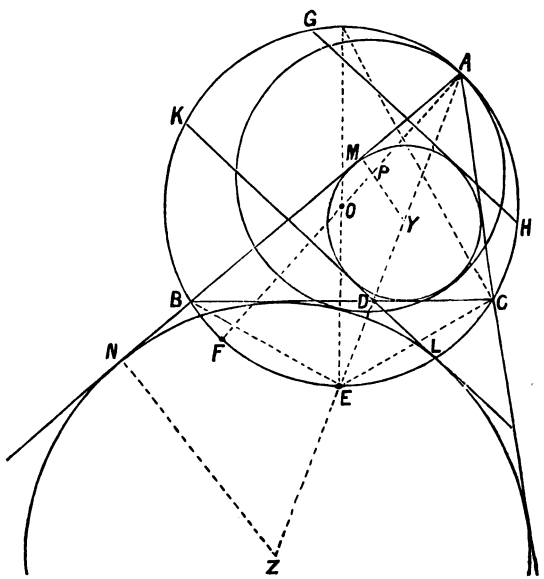
14584. (R. F. Davis, M.A.)—ABC is a given triangle; O circumcentre, R circumradius. If along AO AP be set off = $R\cos^2\frac{1}{2}A$, the circle centre P, radius PA touches the incircle of ABC (as well, of course, as the circumcircle). Hence, if A be a fixed point on a given circle, and PQ a variable chord of constant length, the incircle of the triangle APQ touches a fixed circle. (Due to Mr. C. E. McVicker, M.A.)

Solution by F. L. Ward, B.A.; Lionel E. Reay, B.A.; *and others.*

Let Y be the centre of the incircle and GH the chord of the circumcircle I perpendicular to AO, and touching the incircle. The two circles being so related that a triangle may be inscribed in one and circumscribed to the other, any number of triangles may be so constructed; hence GH is the base of such a triangle. Hence, by a well known



property of the incircle, AG = AH = AY. Therefore the circumcircle is the inverse of GH with respect to the circle centre A and radius AY.

Let the incircle touch AB in M. Since AM = AY cos ½A, therefore the circle centre P and radius PA is the inverse of GH with respect to a circle centre A and radius AM. But, under these conditions, the incircle inverts into itself; and, since GH touches the incircle, the proposed circle must do so also.

The proof of the property above assumed—that, if two circles be so connected that a triangle may be inscribed in one and circumscribed to

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another, an infinite number may be so constructed—may be very simply given as follows:—Let AY meet the circumcircle in E; EC = 2R sin ½A, and r = YA sin ½A; ∴ EC.YA = 2Rr. But EC = EY; ∴ EY.YA = 2Rr.

Now the rectangle EY.YA is constant for every line such as EA drawn through Y; hence, choosing any vertex such as A, and joining A to Y to meet circumcircle in E, a circle with centre E and radius EY meets the circumcircle in points B and C such that Y is the centre of the incircle of the triangle ABC, and its radius, being given by 2Rr, is, therefore, constant.

If, instead of the incircle, we had had the excircle, the proof would have been exactly similar. Let Z be the excentre opposite A. Then EC = EZ = EY, and therefore, just as before, rect. ZE.ZA = 2Rr_a, where r_a is radius of the A excircle.

The circle centre P and radius PA may also be shown to touch the excircle opposite A. Thus, let AO meet circumcircle again in F. Let bisector of ∠A meet BC in D. Drop ZN perpendicular to AB produced. Since ∠FEA is a right angle, and EZ = EY, therefore FZ = FY. Let a circle centre F and radius FY meet the circumcircle in K and L. By the properties above mentioned, and since FZ = FY, KL will be a tangent to both incircle and excircle, and will therefore pass through D. Now AE.AD = $b.c$ = AY.AZ; therefore the circumcircle will invert into KL with respect to the circle centre A and radius $\sqrt{(bc)}$. But AM = AY cos ½A and AN = AZ cos ½A. Therefore the circle centre P and radius PA inverts into KL with respect to the circle centre A and radius $\sqrt{(AM.AN)}$. But the incircle and excircle invert into each other under these conditions; hence the proposed circle touches both the incircle and the excircle.

[Prof. SANJÁNA, who is in substantial agreement with the foregoing, and had also noted that the excircle touching BC has for its envelope the same circle of radius R cos² ½A, refers to the process set forth in his note to Quest. 14014, Vol. LXXI., p. 123.

The PROPOSER adds:—"Mr. MCVICKER combines inversion from A, constant = $s(s-a)$, with subsequent reflexion in the bisector of the angle A; the incircle being thus transformed into the excircle opposite A and *vice versa*. Since BC touches both circles, the above transformation shows that the circle through A whose centre lies on AO (isogonal to the perpendicular from A on BC) and whose diameter = $s(s-a)/(2\Delta/a)$ also touches both circles. On this method of double inversion applied to triangles, see *Mathematical Gazette*, June, 1898. Analytically, if x = PA, the equation

$$x(R^2-2Rr)+(R-x)\,r^2\operatorname{cosec}^2\tfrac{1}{2}A = R(x-r)^2+Rx(R-x)$$

gives x = R cos² ½A. It can be shown similarly that the remaining excircles touch a fixed circle of radius R sin² ½A, using as constant $(s-b)(s-c)$."

13603. (I. ARNOLD.)—A gentleman gets £100,000, which he invests at 4 per cent. compound interest, but spends £8,000 yearly. In what time will he have nothing? (The exact time is required.)

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Solution by W. J. GREENSTREET, M.A.

If x be the number of years, then (SMITH, § 310, iii.)

$$100000 = 8000/_{\frac{1}{25}}\left[1-\left\{1/(1+\tfrac{1}{25})^x\right\}\right], \quad i.e., \quad (104)^x = 2;$$

$$\therefore \quad x = \log 2/\log 1{\cdot}04 = {\cdot}3010.../{\cdot}0170 \ldots = \tfrac{301}{17}$$

$$= 18 \text{ years, 3 months, 2 weeks (to a week).}$$

14680. (Professor SANJÁNA, M.A.)—Prove that the locus of points from which tangents drawn to the ellipse $x^2/a^2 + y^2/b^2 - 1 \equiv S = 0$ form with their chords of contact triangles whose orthocentres lie on S is the sextic

$$x^4y^2a^2(a^2-b^2)^2 + x^2y^4b^2(a^2-b^2)^2 - 2x^2y^2(a^8-a^4b^4+b^8) - x^4a^2b^6 - y^4b^2a^6 + x^2b^6(a^2+b^2)^2 + y^2a^6(a^2+b^2)^2 = 0.$$

Show that this sextic breaks up into the conic $a^2x^2 + b^2y^2 = (a^2+b^2)^2$ and the quartic $b^6x^2 + a^6y^2 = x^2y^2(a^2-b^2)^2$; and explain the result. In the parabola $y^2 = 4ax$ the locus is the quartic

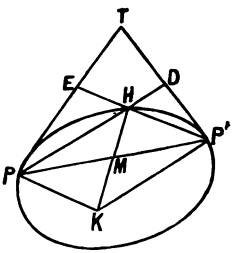
$$(x+2a)(xy^2 + 2ay^2 + 4a^3) = 0.$$

Explain the appearance of the two factors.

Solution by W. J. GREENSTREET, M.A.

Let H be the orthocentre of the triangle TPQ. Parallels through P, P′ to P′H, PH meet in K, and HK cuts PP′ in M. Then, if T, K, H, M be respectively (α, β), (α_1, β_1), (α_2, β_2), (p, q), we have for M

$$p/\alpha = q/\beta = a^2b^2/(b^2\alpha^2 + a^2\beta^2).$$



Also from the relation between the pole and normal pole

$$\frac{\alpha_1}{\alpha} = -\frac{c^2(\beta^2-b^2)}{a^2\beta^2+b^2\alpha^2}, \quad \frac{\beta_1}{\beta} = \frac{c^2(\alpha^2-a^2)}{a^2\beta^2+b^2\alpha^2}.$$

Also $2p = \alpha_1 + \alpha_2$, $2q = \beta_1 + \beta_2$;

therefore

$$\alpha_2(a^2\beta^2 + b^2\alpha^2) = \alpha(b^2a^2 + b^4 + c^2\beta^2),$$

$$\beta_2(a^2\beta^2 + b^2\alpha^2) = \beta(a^2b^2 + a^4 - c^2\alpha^2).$$

But $\alpha_2^2/a^2 + \beta_2^2/b^2 = 1$; therefore on substituting we obtain the sextic as stated. Suppose PP′ a normal chord; H will then be at P or P′. But the locus of the poles of normal chords is $c^4x^2y^2 = a^6y^2 + b^6x^2$; therefore this is part of the locus of T, and therefore must be a portion of the sextic. On trial, the remaining part is found to be the conic

$$a^2x^2 + b^2y^2 = (a^2+b^2)^2,$$

a concentric ellipse with its axes on the axes of the original and of magnitude $(a^2+b^2)/a$, $(a^2+b^2)/b$.

In the parabola, if T is (α, β), H is $-(\alpha+2a)$, $\beta(\alpha+2a)/a$; therefore

$$\beta^2(\alpha+2a)^2 + 4a^3(\alpha+2a) = 0,$$

or locus is $(x+2a)[y^2(x+2a) + 4a^3]$.

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Otherwise.—In the parabola the locus of the poles of normal chords is

$$xy^2 + 2ay^2 + 4a^3 = 0,$$

which is therefore part of the locus, the rest of the quartic being as stated.

6803. (A. J. Allen, B.A.)—If r and s be unequal integers, prove that

$$\Sigma \sec\left(\frac{2r\pi}{n} + \theta\right) \sec\left(\frac{2s\pi}{n} + \theta\right) = \frac{\frac{1}{2}(-n^2)}{1-(-1)^{\frac{1}{2}n}\cos n\theta},$$

if n is even, and $= 0$, if n is odd, the summation extending to all integra values of r and s from 0 to $n-1$ inclusive.

Solution by Professor Sanjána, M.A.

When n is even we have

$$(-1)^{\frac{1}{2}n}\cos n\theta = 1 - \frac{n^2}{2!}\cos^2\theta + \frac{n^2(n^2-2^2)}{4!}\cos^4\theta \ldots + A\cos^n\theta.$$

The summation required being the sum of products of the cosines taken $n-2$ at a time divided by the product of all the cosines, we get this

$$= \frac{\frac{1}{2}(-n^2)}{A} + \frac{1-(-1)^{\frac{1}{2}n}\cos n\theta}{A} = \frac{\frac{1}{2}(-n^2)}{1-(-1)^{\frac{1}{2}n}\cos n\theta},$$

for $$\cos n\left(\frac{2k\pi}{n} + \theta\right) = \cos n\theta.$$

When n is odd we have

$$(-1)^{\frac{1}{2}(n-1)}\cos n\theta = \frac{n}{1!}\cos\theta - \frac{n(n^2-1^2)}{3!}\cos^3\theta - \ldots + B\cos^n\theta;$$

the sum of products of the roots taken $n-2$ at a time being zero, the sum of products of the secants taken two at a time vanishes. See Hobson, *Trigonometry*, ch. vii., series (11) and (14).

14677. (Professor Jan de Vries.)—The sides of three complete $(2n+1)$-laterals are touched by a conic. Prove that their vertices are joined by a curve of degree $3n$.

Solution by the Proposer.

The three sets of $2n+1$ lines determine an involution of degree $3n+1$ formed by the tangents of the conic. To prove this, suppose the involution represented by its image on a line. Then, z being the abscissa of a point of this line, the new involution is determined by an equation of the form

$$f_{2n+1}(z)\,\phi_n(z) = \lambda F_{2n+1}(z)\,\psi_n(z).$$

Here f and F are known functions of degree $2n+1$, whilst the coefficients of ϕ and ψ must be found from the third set of $2n+1$ points.

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Substituting the parameters of these $2n+1$ points in the above equation, we get a set of $2n+1$ linear equations with $2n+1$ variables, viz., the coefficients of ϕ, the parameter λ, and the coefficients of ψ, each multiplied by λ.

Our involution, therefore, is determined by the given three groups of $2n+1$ elements. Now, if we observe that the locus of intersections of the lines of each group formed by $3n+1$ tangents of the conic is a curve of degree $3n$, the proposition is proved.

14593. (Robert W. D. Christie.)—Find *all* the primitive roots of any prime of the form $12l+1$ from a *single* equation

$$ax^{2k} \pm bx^k - c = 0 \bmod 12l+1,$$

and extend the same, if possible, to all primes.

E.g., $x^2 \pm 5x - 1 \equiv 0 \pmod{13}$.

Solution by the Proposer.

By the doctrine of primitive roots, we have

$$r^m = \pm \frac{2xy}{x^2-y^2} \bmod p_{4n+1} \text{ or } \left\{\frac{2\,(1/x\,.\,1/y)}{(1/x)^2-(1/y)^2}\right\},$$

also $r^{2m} - r^m + 1 = 0 \bmod P_{6m+1}$ (see Quest. 14153).

Combining these two modular equations, we get

$$zr^{2l} + r^l(x^2-y^2) - z = 0 \bmod 12l+1, \quad \text{where } z = p - 2xy.$$

E.g., $13 = 3^2+2^2$; therefore $z = 1$ and $l = -1$; thus $r^2 \pm 5r - 1 = 0 \bmod 13$.

$73 = 8^2+3^2$; therefore $z = 73-48 = 25$, $l = 6$; therefore

$$25r^{12} \pm 55r^6 - 25 = 0 \bmod 73;$$

i.e., $5r^{12} \pm 11r^6 - 5 = 0 \bmod 73$ or $5r^{18} \pm 11 = 0 \bmod 73$.

If $z = p - 2xy = 1$, then $x - y = 1$; therefore

$$r^{2l} + (x+y)\,r^l - 1 = 0 \bmod 12l+1.$$

14469. (H. MacColl, B.A.)—Professor Savage's Quest. 14394 has suggested the following:—Given that a is real and positive, and that x and y are each taken at random between a and $-a$, what is the chance that $(x+y)^n$ is less, and $(x+y)^{n+1}$ greater, than a? Show that there are four different answers (including *zero*) depending on the oddness or evenness of n, and on the numerical limits assigned to a.

Solution by the Proposer.

Let A denote our data that x and y are each taken at random between a and $-a$; let Q assert that $(x+y)^n - a$ is negative; and let R assert that $(x+y)^{n+1} - a$ is positive. The chance required is a fraction the numerator of which is the integral $\int dx \int dy$ subject to the restriction QRA, and of

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which the denominator is the same integral subject only to the restriction A. The only difficulty is to find the limits of integration from the statement QRA. These may be found by my Calculus of Limits, for examples of which see *Reprint*, Vol. LXXIII., p. 69, with Note ii., p. 70, and Vol. LXXIV., pp. 49 and 59. My results are the following :—

(1) When n is even, and $a \not> 1$, the chance of QR is *zero*.

(2) When n is even, and $a > 1$, the chance of QR is

$$(a^{1/n} - a^{1/(n+1)})\,(4a - a^{1/n} - a^{1/(n+1)})/8a^2.$$

(3) When n is odd, and $a < 1/2^{(n+1)/n}$, the chance of QR is *zero*.

(4) When n is odd, and a between 1 and $1/2^{(n+1)/n}$, the chance of QR is

$$(2a - a^{1/(n+1)})^2/8a^2.$$

(5) When n is odd, and $a > 1$, the chance of QR is

$$\left\{2\,(a^{1/n} - a^{1/(n+1)})\,(2a - a^{1/n}) + (a^{1/n} - a^{1/(n+1)})^2 + (2a - a^{1/(n+1)})^2\right\}/8a^2.$$

Note.—A full solution of this question will be found in my memoir on "La Logique Symbolique et ses Applications" in Vol. III. of *La Bibliothèque du Congrès International de Philosophie*.

6747. (A. J. C. ALLEN, B.A.)—A point P is joined to K, the orthocentre, and the line PK produced to meet the sides BC, CA, AB of a triangle ABC in D, E, F; prove (1) that, if

$$\frac{PE \cdot PF}{KE \cdot KF}\sin 2A + \frac{PF \cdot FD}{KF \cdot KD}\sin 2B + \frac{PD \cdot PE}{KD \cdot KE}\sin 2C = \text{a const.},$$

the locus of P is a circle; and (2) find its centre and radius.

Solution by Professor SANJÁNA, M.A.

If AK, BK, CK meet the opposite sides in X, Y, Z respectively, and α, β, γ be the trilinear coordinates of P, we have

$$\frac{PD}{KD} = \frac{\alpha}{KX} = \frac{\alpha}{2R\cos B\cos C}, \quad \frac{PE}{KE} = \frac{\beta}{2R\cos C\cos A},$$

$$\frac{PF}{KF} = \frac{\gamma}{2R\cos A\cos B}.$$

Thus the locus of P is readily seen to be

$$\beta\gamma\sin A + \gamma\alpha\sin B + \alpha\beta\sin C = k \cdot 2R^2\cos A\cos B\cos C,$$

a circle concentric with the circumcircle. Write this in the form

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = k \cdot 4R^3\cos A\cos B\cos C;$$

also, taking ρ to be its radius, we have for the equation

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = (\rho^2 - R^2)(2\Delta)^2 + abc$$

(C. SMITH, *Conics*, xiii., Ex. 8). Comparing, we get

$$\rho^2 = R^2\left\{1 + kabcR\cos A\cos B\cos C + \Delta^2\right\},$$

or

$$\rho = R\left\{1 + 2k\cot A\cot B\cot C\right\}^{\frac{1}{2}}.$$

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Note to Quest. 6747. By Professor SANJÁNA, M.A.

When I wrote out the solution of this question, I did not know that it was the same as 6085, which is solved in Vol. LXXII., by Mr. J. GILBART SMYLY. His results can all be obtained very simply by employing my methods. If the point O be fgh, we have PD + OD = a/f, &c. Hence the locus of P is

$$k(a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2) + \beta\gamma(2kbc - 4\Delta^2 L/gh) + \ldots + \ldots = 0.$$

The condition for a circle gives

$$L\,be/gh = M\,ea/hf = N\,ab/fg, \quad \text{or} \quad L : M : N = agh : bhf : efg.$$

Thus, if O be the orthocentre,

$$L : M : N = a\cos A : b\cos B : c\cos C = \sin 2A : \sin 2B : \sin 2C;$$

the equation of the circle then becomes $\Sigma a\beta\gamma = 4kR^3 \cos A \cos B \cos C$, and its radius $R\{1 + 2k \cot A \cot B \cot C\}^{\frac{1}{2}}$. [Mr. SMYLY's expression for the radius seems incorrect.] For the circumcentre

$$L : M : N = a/\cos A : b/\cos B : c/\cos C = \tan A : \tan B : \tan C.$$

For the centroid, the symmedian point, the incentre, the ratios are

$$\sin^2 A : \sin^2 B : \sin^2 C, \quad 1 : 1 : 1, \quad \sin A : \sin B : \sin C,$$

respectively. For two isogonally conjugate points, the products of the corresponding ratios are $\sin^2 A$, $\sin^2 B$, $\sin^2 C$.

6729. (A. J. C. ALLEN, B.A.)—If the centre I of the inscribed circle of a triangle be joined to the centroid G, and if IG be produced to G′, so that IG′ = 2IG, show that the inscribed circle will pass through G′ if

$$3\sin\tfrac{1}{2}A \sin\tfrac{1}{2}B \sin\tfrac{1}{2}C(5\sin\tfrac{1}{2}A \sin\tfrac{1}{2}B \sin\tfrac{1}{2}C - 4) + 1 + \cos A \cos B \cos C = 0.$$

Solution by Professor SANJÁNA, M.A.

Let Σ_1 stand for $\cos A + \cos B + \cos C$, Σ' for $\cos B \cos C + \cos C \cos A + \cos A \cos B$, and Π for $\cos A \cos B \cos C$. Then the condition to be proved, which is

$$15\sin^2\tfrac{1}{2}A \sin^2\tfrac{1}{2}B \sin^2\tfrac{1}{2}C - 12\sin\tfrac{1}{2}A \sin\tfrac{1}{2}B \sin\tfrac{1}{2}C + 1 + \cos A \cos B \cos C = 0$$

or $15(1-\cos A)(1-\cos B)(1-\cos C) - 24(\cos A + \cos B + \cos C - 1) + 8 + 8\cos A \cos B \cos C = 0,$

becomes $$47 - 39\Sigma_1 + 15\Sigma' - 7\Pi = 0.$$

Now the question requires $r = IG' = 2IG$, *i.e.* (squaring),

$$16R^2 \sin^2\tfrac{1}{2}A \sin^2\tfrac{1}{2}B \sin^2\tfrac{1}{2}C = \tfrac{9}{4}R^2\{\Sigma \sin^2\tfrac{1}{2}B \sin^2\tfrac{1}{2}C - \tfrac{1}{16}\Sigma \sin^2 A\};$$

see Quest. 5683, Vol. LXX. This evidently gives

$$(1-\cos A)(1-\cos B)(1-\cos C) = \tfrac{9}{8}\Sigma(1-\cos B)(1-\cos C) - \tfrac{9}{8}(\sin^2 A + \sin^2 B + \sin^2 C);$$

therefore $$1 - \Sigma_1 + \Sigma' - \Pi = 8 - \tfrac{1}{8}\Sigma_1 + \tfrac{9}{8}\Sigma' - \tfrac{1}{64}\Sigma_1 - \tfrac{1}{64}\Pi,$$

or $47 - 39\Sigma_1 + 15\Sigma' - 7\Pi = 0$, which proves the theorem.

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14588. (R. P. PARANJPYE, B.A.)—Prove by elementary considerations, and without the use of continued fractions, that the most general positive integral solution of the equation $x^2-2y^2=z^2$ is

$$x=k(\lambda^2+2\mu^2),\quad y=2k\lambda\mu,\quad z=\pm k(\lambda^2-2\mu^2),$$

where λ and μ are prime to each other, and the ambiguous sign is chosen to make z positive.

Note by ALETROP.

L'équation de l'énoncé c'est la bien connue équation de PELL (pour une notice bibliographique, *Journal de Mathématiques élémentaires*, de LONGCHAMPS, 1884, pp. 15 et 19).

Voici une solution bien simple de l'équation plus générale

$$ax^2-by^2=cz^2 \quad \text{.....(i.).}$$

En posant $x=kx',\quad y=ky',\quad z=kz'$ .....(ii.),

il vient $ax'^2-by'^2=cz'^2$, qui peut s'écrire

$$(\sqrt{a}\,x'-\sqrt{b}\,y')/z'^2=c/(\sqrt{a}\,x'+\sqrt{b}\,y'),$$

d'où, μ étant une indéterminée, $\sqrt{a}\,x'-\sqrt{b}\,y'=c\mu,\ z'^2=(\sqrt{a}\,x'+\sqrt{b}\,y')\mu$. De ces deux équations on tire x', y' en fonction de z', dont la valeur étant quelconque nous la représenterons par λ; puis, en égard de (ii.),

$$x=\left\{k(\lambda^2+c\mu^2)\right\}/2\sqrt{a}\,\mu,\quad y=\left\{k(\lambda^2-c\mu^2)\right\}/2\sqrt{b}\,\mu,\quad z=k\lambda.$$

Les solutions entières de (i.) seront

$$x=k\sqrt{b}(\lambda^2+c\mu^2),\quad y=k\sqrt{a}(\lambda^2-c\mu^2),\quad z=2k\sqrt{(ab)}\,\lambda\mu.$$

De cettes-ci on deduit comme cas particulier les solutions générales entières et positives de l'équation de PELL.

Solution by the PROPOSER.

$$x^2-2y^2=z^2;\quad \therefore\quad x^2-z^2=2y^2.$$

Now the difference of two squares being only of the form $4m$, $4m\pm1$, y is an even integer, say 2ρ;

$$\therefore\quad x^2-z^2=8\rho^2;\qquad \therefore\quad (x-z)(x+z)=8\rho^2.$$

Supposing now x, y, z have no common factor, it follows that either

$$x-z=2\lambda^2,\quad x+z=4\mu^2,\quad \rho=\lambda\mu,$$

or

$$x+z=2\lambda'^2,\quad x-z=4\mu'^2,\quad \rho=\lambda'\mu',$$

λ and μ being prime to each other. If not, then $x-z$ and $x+z$ will have a factor common other than 2, and then x, z, and therefore y, will not be prime to each other.

Either of these hypotheses gives the form in the enunciation on making provision for any common factor that x, y, z may have.

Further Note on Quest. 14588. *By* ALETROP.

On sait, et il est demontré par des considérations élémentaires (*Giornale di Matematiche*, de BATTAGLINI, t. VII., 1869, p. 313), que les solutions *générales* de l'équation $x^2=y_1^2+y_2^2+y_3^2+\ldots+y_{n+1}^2$

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sont $x = k(a_1^2 + a_2^2 + a_3^2 + \ldots + a_{n+1}^2)$, $y_1 = 2ka_1a_{n+1}$, $y_2 = 2ka_2a_{n+1}$, $y_3 = 2ka_3a_{n+1}$, ..., $y_n = 2ka_na_{n+1}$, $y_{n+1} = k(a_1^2 + a_2^2 + a_3^2 + \ldots - a_{n+1}^2)$, où k, a_1, a_2, a_3, ..., a_{n+1} désignent des arbitraires. Donc en faisant $y_1 = y_2 = y_3 = \ldots = y_n$ et $y_{n+1} = z$, et écrivant μ au lieu de $a_1 = a_2 = a_3 = \ldots = a_n$ et λ au lieu de a_{n+1}, on a

$$x = k(n\mu^2 + \lambda^2), \quad y = 2k\mu\lambda, \quad z = k(n\mu^2 - \lambda^2)$$

pour des solutions générales de l'équation $x^2 - ny^2 = z^2$.

La question proposé répond au cas de $n = 2$.

14559. (Lt.-Col. ALLAN CUNNINGHAM, R.E.)—Solve

$$s_x = s_\alpha . s_y, \quad s_\alpha . s_x = s_\alpha . s_y, \quad s_\alpha . s_\beta . s_\gamma \ldots s_x = s_\alpha . s_b . s_c \ldots s_y,$$

where s denotes a sum of *successive* integer cubes.

Solution by the PROPOSER.

Let $$s_n = 1^3 + 2^3 + 3^3 + \ldots + n^3 = \{\tfrac{1}{2}n(n+1)\}^2 = T_n^2,$$

where T_n denotes a *triangular*.

Here α, β, γ, ... ; a, b, c, ... are supposed *given* integers ; x, y the unknowns. The solutions of $T_x = T_\alpha . T_y$, $T_\alpha . T_x = T_\alpha . T_y$, &c., are solutions of $s_x = s_\alpha . s_y$, $s_\alpha . s_x = s_\alpha . s_y$, &c. Now

$$T_x = \tfrac{1}{2}x(x+1) = \tfrac{1}{8}(X^2 - 1), \quad T_y = \tfrac{1}{2}y(y+1) = \tfrac{1}{8}(Y^2 - 1) \ldots\ldots (1),$$

where $$X = 2x + 1, \quad Y = 2y + 1 \ldots\ldots\ldots\ldots (2).$$

(i.) $s_x = s_\alpha . s_y$ gives, on reduction, $x^2 - 1 = T_\alpha (Y^2 - 1)$; whence

$$X^2 - T_\alpha . Y^2 = 1 - T_\alpha \ldots\ldots\ldots\ldots (3).$$

Now, let ξ, η be the *least* solutions of the auxiliary equation $\xi^2 - T_\alpha . \eta^2 = 1$ (which is always possible). Then the solutions (X_r, Y_r) of the Diophantine (3) are derivable *in succession* by the relations

$$X_r = \xi X_{r-1} \mp T_\alpha . \eta Y_{r-1}, \quad Y_r = \xi Y_{r-1} \mp \eta X_{r-1} \ldots\ldots\ldots (4),$$

beginning from the obvious solution $X_1 = 1$, $Y_1 = 1$; and x, y are then given by (2).

Ex.—Let $\alpha = 2$; then $T_\alpha = 3$: and the successive values are

$(X, Y) = (1, 1), (5, 3), (19, 11), (71, 41), (265, 153)$, &c.,

$(x, y) = (\ .\ .\ ), (2, 1), (9, 5), (35, 20), (132, 76)$, &c.

(ii.) $s_\alpha . s_x = s_\alpha . s_y$ gives, on reduction, $T_\alpha (X^2 - 1) = T_\alpha (Y^2 - 1)$; whence

$$T_\alpha . X^2 - T_\alpha . Y^2 = T_\alpha - T_\alpha \ldots\ldots\ldots\ldots (5).$$

Now, let ξ, η be the *least* solutions of the auxiliary equation

$$\xi^2 - T_\alpha T_\alpha . \eta^2 = 1$$

(which is always possible). Then the solutions (X_r, Y_r) of the Diophantine (5) are derivable *in succession* by the relations

$$X_r = \xi X_{r-1} \pm T_\alpha . \eta Y_{r-1}, \quad Y_r = T_\alpha . \eta X_{r-1} \pm \xi Y_{r-1} \ldots\ldots (6),$$

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beginning from the obvious solution $X_1 = 1$, $Y_1 = 1$; and x, y are then given by (2).

Ex.—Let $a = 2$, $a = 4$; then $T_a = 3$, $T_a = 10$; and the successive values are (X, Y) = (1, 1), (9, 5), (31, 17), (199, 109), &c.,

$(x, y) = (\ .\ .\ )$, (4, 2), (15, 8), (99, 54), &c.

(iii.) $s_\alpha \,.\, s_\beta \,.\, s_\gamma \ldots s_x = s_a \,.\, s_b \,.\, s_c \ldots s_y$ may be solved by a process quite similar to that of Problem (ii.); but numerical examples would run into very high numbers unless α, β, γ, &c.; a, b, c, &c. be such that $(s_\alpha \,.\, s_\beta \,.\, s_\gamma \ldots) \div (s_a \,.\, s_b \,.\, s_c \ldots)$ reduces to a simple ratio.

14645. (W. R. Hughes.)—PP′, PM are diameters of an ellipse and a circle, osculating at P, and MP′ meets the ellipse again in Q. Prove that the tangents at M and Q and the common chord PR meet in a point.

Solution by Constance I. Marks, B.A.

Take the normal and tangent at P as Ox, Oy.

Then the equations to the ellipse and the circle may respectively be written in the forms:

$$ax^2 + 2bxy + cy^2 + 2dx = 0 \quad \ldots\ldots \text{(i.)},$$

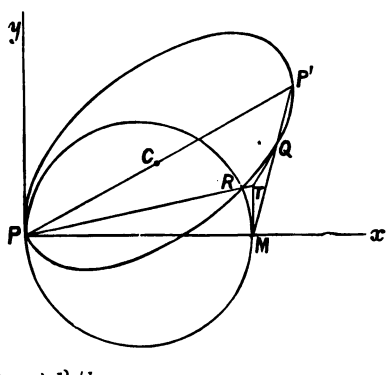
$$cx^2 + cy^2 + 2dx = 0 \quad \text{(ii.)}.$$

Hence equation to common chord PR is

$$(a-c)x + 2by = 0 \quad \text{(iii.)},$$

and equation to tangent to (ii.) at M $\{-(2d/c),\ 0\}$ is

$$cx + 2d = 0 \quad \ldots \text{(iv.)}.$$



From (iii.) and (iv.), coordinates of T are $-(2d/c)$, $\{(a-c)\,d\}/bc$.

Again, if C is the centre of the ellipse, the coordinates of C are given by

$$ax + by + d = 0, \quad bx + cy = 0 \quad \ldots\ldots\ldots\ldots \text{(v.)},$$

whence $x = (-cd)/(ac-b^2)$, $y = bd/(ac-b^2)$, at C;

and therefore $x = (-2cd)/(ac-b^2)$, $y = 2bd/(ac-b^2)$, at P′.

Therefore equation to P′M is

$$\{y(ac-b^2)\}/bd = (x + 2d/c)/\{(-cd)/(ac-b^2) + d/c\};$$

that is, $y/b = (cx + 2d)/(-c^2 + ac - b^2)$ or $bcx + (c^2 - ac + b^2)\,y = -2bd$...(vi.).

Now the lines joining P to the intersections of (i.) and (vi.) are

$$ax^2 + 2bxy + cy^2 + x\left[-\{bcx + (c^2 - ac + b^2)\,y\}/b\right] = 0;$$

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that is, $(ab-bc)\,x^2+(b^2+ac-c^2)\,xy+bcy^2=0$;

that is, $y^2/x^2+(b^2+ac-c^2)/bc\,.\,y/x+(ab-bc)/bc=0$,

or $(y/x+b/c)\{y/x+(a-c)/b\}=0$ ..... (vii.).

Since PP′ is the line $y/x+b/c=0$, therefore equation to PQ is

$$y/x+(a-c)/b=0 \quad \text{..... (viii.).}$$

Therefore at Q we must have

$$ax^2+2x^2(c-a)+\{cx^2(a-c)^2\}/b^2+2dx=0,$$

and $x \neq 0$; therefore

$$x=2b^2d/(ab^2-2b^2c-a^2c+2ac^2-c^3) \quad \text{..... (ix.),}$$

and therefore $y=\{-2bd(a-c)\}/(ab^2-2b^2c-a^2c+2ac^2-c^3)$.

The polar of T with respect to (i.) is

$$-(2ad/c)\,x+x\{(a-c)/c\}\,d-by\,(2d/c)+cy\{(a-c)/bc\}\,d+dx-2d^2/c=0;$$

$$x(-ad/c)+y[-(2bd/c)+\{(a-c)/b\}\,d]-2d^2/c=0.$$

It is easily verified that this line cuts (viii.), where x has the value given by (ix.), or TQ is the tangent at Q to (i.).

A Method of Solving certain Series of the Form $(Ay^n+Bny^{n-1}\ldots+)+(A_1y_1^n+B_1ny_1^{n-1}\ldots+)$, &c., *$y$ and y_1, &c., being integers.*

By Samuel C. Gunning, J.P. (Cookstown).

Let u_0, u_1, u_2, u_3, &c., be the terms of any series (S_0); the $(n+1)$-th term is u_n. Differentiating this as under,

u_0, u_1, u_2, u_3, u_4, &c.,

$(\pm u_0+u_1)$, $(\pm u_1+u_2)$, $(\pm u_2+u_3)$, $(\pm u_3+u_4)$, &c.,

$(u_0\pm 2u_1+u_2)$, $(u_1\pm 2u_2+u_3)$, &c.,

$(\pm u_0+3u_1\pm 3u_2+u_3)$, &c.,

&c.,

we get a series of first terms (differences) the $(n+1)$-th term of which may be denoted by $(\pm 1+u)_n$. Performing the same operation on the series represented by $(\pm 1+u)_n$, we get one of which the general term is in like manner $(\pm 2+u)_n$, and from $(\pm 2+u)_n$ we get $(\pm 3+u)_n$, and so on; each operation, as may be shown, only adds unity to the formula. Therefore the $(n+1)$-th term of all these series may be given as $(y+u)_n$, the expansion of which is

$$y^nu_0+ny^{n-1}u_1+\tfrac{1}{2}[n(n-1)]\,y^{n-2}u_2+\&c.,$$

y being the number of operations, and positive or negative according to procedure, the successive values of u being the terms of S_0 taken in order.

Therefore we get the general term of a series if, on subjecting it to the operations here indicated, we finally reach a series the terms of which can be readily substituted for the values of u in the corresponding

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terms of the formula. For example, required the $(n+1)$-th term of the series 2, 3, 7, 22, 72, 224, &c.,

2	**3**	**7**	**22**	**72**	**224**
	1	4	15	50	152
		3	11	35	102
			8	24	67
				16	43
					27
2	**1**	**3**	**8**	**16**	**27**
	−1	2	5	8	11
		3	3	3	3
			0	0	0

This terminates with three terms. Substituting them in the formula, we get $2y^n - ny^{n-1} + \frac{1}{2}[3n(n-1)]y^{n-2}$,

and, y being $= 2$, we have, on simplifying, the required term

$$= (3n^2 - 7n + 16)\, 2^{n-3}.$$

The general terms thus obtained can be integrated at once by the method of finite differences; but the sum may also be obtained by this method, for the $(n+1)$-th term of a series minus its first term is the sum to n terms of the series derived from it as a first order by subtraction, thus

$$u_0,\ u_1,\ u_2,\ u_3,\ \ldots,\ u_n,$$

$$(u_1 - u_0),\ (u_2 - u_1),\ (u_3 - u_2),\ \ldots,\ (u_n - u_{n-1}).$$

It is evident that the sum to n terms of the first order is $u_n - u_0$; therefore we get the sum of a series by finding the $(n+1)$-th term of the ungiven series from which it is so derived. We may assume 0 as the first term of the ungiven series, and any irregularity caused by this assumption can be readily removed as afterwards explained. For example, required the sum to n terms of the series $-2-6-10+18+270$, &c.

0	.....			
	−2	**−6**	**−10**	**18**	**270**
		−4	**4**	28	252
			0	32	224
				32	192
					160
0	**−2**	**−4**	0	32	160
	−2	**−2**	4	32	128
		0	6	28	96
			6	22	68
				16	46
					30
0	**−2**	0	6	16	30
	−2	2	6	10	14
		4	4	4	4
			0	0	0

Substituting, we get $-2ny^{n-1} + \frac{1}{2}[4n(n-1)]y^{n-2}$,

and simplifying, y being $= 3$,

$$S = 2(n^2 - 4n)\, 3^{n-2}.$$

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An irregularity in either the differences or an order is due to the presence or absence of a multiple figurate series, and must be removed before proceeding further with the operation. Its value in making up the required term is the general term of the figurate series found by this method. Let t be the irregularity, and let it occur in the fourth term of the differences and in the second term of the third order. It arises in this way:

In the Differences.	*In an Order.*
0 0 0 t $4t$ $10t$ $20t$	0 0 0 0 t $3t$ $6t$
0 0 t $3t$ $6t$ $10t$	0 0 0 t $2t$ $3t$
0 t $2t$ $3t$ $4t$	0 0 t t t
t t t t	0 t 0 0
0 0 0	

The value of t in the differences is, as in the preceding examples, found at once by substitution in the corresponding term of the formula; but the figurate series which gives rise to t in an order is not quite regular in its early terms, and must be amended. The amendments are differences. In dealing with all these due regard must be had to the varying value of y.

A few Examples.

Required the $(n+1)$-th term of 1, 6, 16, 38, 88, 202, &c.

1	6	16	38	88	202
	5	10	22	50	114
		5	12	28	64
			7	16	36
				9	20
					11

Amending these differences, we have

1	3	5	7	9	11
	2	2	2	2	2
		0	0	0	0

Substituting and adding the value of the alteration, which, having occurred in the previous operation, makes the value of y less by unity, we have

$$y^n + 2ny^{n-1} + 2n(y-1)^{n-1}.$$

Simplifying, the required term is $(n+1)2^n + 2n$.

Required the sum of preceding series to n terms.

0	.....				
	1	6	16	38	88
		5	10	22	50
			5	12	28
				7	16
					9

Amending the differences, we have

−1	1	3	5	7	9
	2	2	2	2	2
		0	0	0	0

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Substituting and adding value of alterations, we have

$$-y^n + 2ny^{n-1} + (y-1)^n + \tfrac{1}{2}[2n(n-1)](y-1)^{n-2},$$
$$S = (n-1)2^n + n^2 - n + 1.$$

Required the sum to n terms of 2, 3, 7, 22, &c.

0	.....				
	2	3	7	22	72
		1	4	15	50
			3	11	35
				8	24
					16
0	2	1	3	8	16
	2	−1	2	5	8

Amending this order, we have

0	2	1	3	8	16
	−4	−1	2	5	8
		3	3	3	3
			0	0	0

Substituting, we get $-4ny^{n-1} + \tfrac{1}{2}[3n(n-1)]y^{n-2}$.

The correction is represented by 6 times a figurate series of the first order, viz., 6 (0, 1, 1, 1, &c.), the general term of which is $6y^n - 6(y-1)^n$. Adding this to portion already found, we have

$$-4ny^{n-1} + \tfrac{1}{2}[3n(n-1)]y^{n-2} + 6y^n - 6(y-1)^n,$$
$$S = (6-2n)2^n + 3n(n-1)2^{n-3} - 6.$$

Required the sum of $1-4+12-32+80$, &c. We have to find the series from which this is derived by *subtraction*, and then proceed by *addition*, thus,

	1	−4	+12	−32	+80	−
0	1	−3	+ 9	−23	+57	= the ungiven series
	1	−2	+ 6	−14	+34	
		−1	+ 4	− 8	+20	
			3	− 4	+12	
				− 1	+ 8	
					7	
0	1	−1	+ 3	− 1	+ 7	
	1	0	+ 2	+ 2	+ 6	
		1	+ 2	+ 4	+ 8	
			3	+ 6	+12	
				9	+18	
					27	

Amending the differences, we have

$$\tfrac{1}{6},\ \tfrac{1}{3},\ 1,\ 3,\ 9,\ 27,\ \text{&c.},$$

which, substituted in the formula, give

$$\tfrac{1}{6}[(y+3)^n].$$

Adding to this for the alterations, we get

$$\tfrac{1}{6}(y+3)^n - \tfrac{1}{3}y^n + \tfrac{1}{3}(2ny^{n-1}),$$

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and, y being $= -2$, we have

$$S = \tfrac{1}{9}[1-(3n+1)(-2)^n].$$

Where the second or other differences show some regularity they may be utilized as in the following example:—

Required the $(n+1)$-th term of 1, 3, 12, 54, 248, 1128, &c.

1	3	12	54	248	1128
	2	9	42	194	880
		7	33	152	686
			26	119	534
				93	415
					322
1	2	7	26	93	322
	1	5	19	67	229
		4	14	48	162
			10	34	114
				24	80
					56

1	1	4	10	24	56	$= S_1$
	0	3	6	14	32	
		3	3	8	18	
			0	5	10	
				5	5	
					0	

Using the third differences, of which the $(n+1)$-th term is $4y^n + 2ny^{n-1}$, and y being $= 2$, and making $n = -2-1, 0, 1, 2$, &c., we get the following series as an amendment of S_1:—

$$\tfrac{1}{2} \quad \tfrac{3}{2} \quad 4 \quad 10 \quad 24 \quad 56,$$

of which the general term is

$$\tfrac{1}{2}y^n + \tfrac{1}{2}(ny^{n-1}).$$

Adding for the alterations, we have

$$\tfrac{1}{2}y^n + \tfrac{1}{2}(ny^{n-1}) + \tfrac{1}{2}(y-2)^n - \tfrac{1}{2}n\,(y-2)^{n-1},$$

and, y being $= 4$, the required term is

$$\tfrac{1}{2}[(n+4)\,4^{n-1} - (n-2)\,2^{n-1}].$$

The sum of a combination of a whole number series with the numerical coefficients of a binomial can be readily obtained by this method; for the sum of the differences is the general term of the series from which the whole number series is derived. Thus, for example:

Required the value of

$$1 + 6n + \frac{16n\,(n-1)}{2} + 38\,\frac{n\,(n-1)(n-2)}{2\,.\,3}, \text{ &c.}$$

We have seen already that the general term of this integer series is

$$y^n + 2ny^{n-1} + 2n\,(y-1)^{n-1},$$

and, as the value of the general term of the series from which the given series is derived is required, we have $y = 3$, and the required summation is

$$(2n+3)\,3^{n-1} + n\,.\,2^n.$$

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From the foregoing it is evident that where only a few initial terms of a numerical series are given there may be constructed an infinite number of different series which shall each have the same initial terms; but, when in dealing with these terms a law of succession appears, it may be assumed that the lowest values of n are required. In propounding all such questions it would be well to give a distant term, so as to secure a definite answer.

14586. (R. C. Archibald, M.A., Ph.D.)—Any circle through the cusp and centre of the base of a cardioid cuts the curve again in two points P_1, P_2, and its double tangent in P_3, P_4. Show that, (1) the line P_1P_2 produced is tangent to the cardioid; (2) the lines P_1P_3, P_2P_4 (which do not meet unless produced) are tangent to the cardioid at P_1, P_2 respectively. Conversely, if any tangent to a cardioid meet the curve again in P_1, P_2, and if the tangents to the curve at P_1, P_2 meet the double tangent in P_3, P_4, the points P_1, P_2, P_3, P_4 lie on a circle through the cusp and centre of the base of the cardioid.

[*Note.*—The tangents P_1P_3, P_2P_4 when produced cut on a hyperbola, and the points P_3, P_4 divide the points of contact of the double tangent harmonically. This result was first stated by Siebeck, *Crelle's Journal*, 1866, p. 362, and later by Wolstenholme in *Educational Times Reprint*, 1874, Vol. xx., p. 34, Quest. 4142.]

Solution by Professor Sanjána, M.A.

Let K be the cusp, C the centre of base, KB ($= a$) the diameter of base; and let KD be the diameter of any circle through K and C. If $\angle DKC = \phi$, and $\tan\phi = t'$, the equation of the circle is

$$r = \tfrac{1}{2}a(\cos\theta + \sin\theta . t').$$

The equation of the cardioid is

$$r = a(1+\cos\theta);$$

hence, at P_1 and P_2, the points of intersection,

$$\sin\theta . t' = 2 + \cos\theta;$$

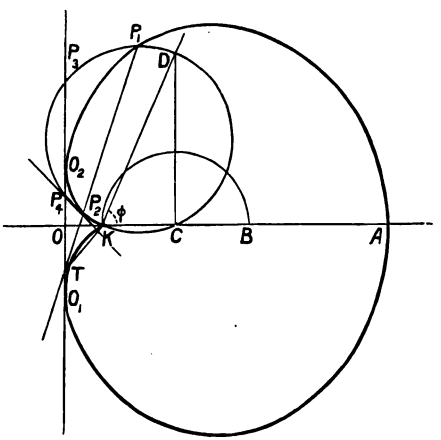
whence $2t . t' = 3 + t^2$,

where $t = \tan\tfrac{1}{2}\theta$.

If Θ be the vectorial angle of any point on the cardioid, and

$$T = \tan\tfrac{1}{2}\Theta,$$

we have $x = 2a(1-T^2)/(1+T^2)^2$, $y = 4aT/(1+T^2)^2$.



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Hence the equation of the line joining the points T_1, T_2 is

$$\frac{y(1+T_1^2)^2-4aT_1}{x(1+T_1^2)^2-2a(1-T_1^2)} = \frac{2}{T_1+T_2}\ \frac{-1+2T_1T_2+T_1^2T_2^2+T_1T_2(T_1^2+T_2^2)}{3+T_1^2+T_2^2-T_1^2T_2^2};$$

and the equation of the tangent at T is

$$y(3T-T^3)+x(1-3T^2)-2a=0.$$

If the points P_1, P_2 be t_1, t_2 respectively, we have $t_1+t_2=2t'$, $t_1t_2=3$; whence the equation of P_1P_2 is

$$\frac{y(1+t_1^2)^2-4at_1}{x(1+t_1^2)^2-2a(1-t_1^2)} = \frac{1}{t'}\ \frac{-4+12t'^2}{-12+4t'^2} = \frac{3t'^2-1}{t'^3-3t'};$$

that is, $y(1+t_1^2)^2(t'^3-3t')+x(1+t_1^2)^2(1-3t'^2)$

$$\begin{aligned}&= 2a\left\{2t_1t'(t'^2-3)+(1-t_1^2)(1-3t'^2)\right\}\\ &= 2a\left\{(t_1^2+3)(t'^2-3)+(t_1^2-1)(3t'^2-1)\right\}\\ &= 2a\left\{4t_1^2t'^2-4t_1^2-8\right\} = 2a\left\{4t_1^2t'^2-4(2t_1t'-3)-8\right\}\\ &= 2a(2t_1t'-2)^2 = 2a(1+t_1^2)^2;\end{aligned}$$

remembering that t_1 satisfies the equation $2t_1t'=t_1^2+3$. Thus P_1P_2 is the line

$$y(t'^3-3t')+x(1-3t'^2)-2a=0,$$

which proves that it touches the curve at $T=-t'=\tan(-\phi)$, so that $\Theta=-2\phi$, or $\angle AKT = 2AKD$.

Combining the double tangent, $4x+a=0$, with the circle KCD, we get

$$16y^2-8yat'+3a^2=0;$$

whence

$$y=\tfrac{1}{4}a\left\{t'\pm\sqrt{(t'^2-3)}\right\}=\tfrac{1}{4}at_1 \text{ or } \tfrac{1}{4}at_2.$$

Thus the coordinates of P_3 are $-\frac{1}{4}a$, $\frac{1}{4}at_2$; those of P_4 are $-\frac{1}{4}a$, $\frac{1}{4}at_1$. Therefore the equation of P_1P_3 is

$$\frac{y-\frac{1}{4}at_2}{x+\frac{1}{4}a} = \frac{4at_1/(1+t_1^2)^2-\frac{1}{4}at_2}{2a(1-t_1^2)/(1+t_1^2)^2+\frac{1}{4}a},$$

or

$$\frac{4y-at_2}{4x+a} = \frac{16t_1-3(1+t_1^2)^2-t_1}{8(1-t_1^2)+(1+t_1^2)^2} = \frac{10t_1^2-3-3t_1^4}{t_1(9-6t_1^2+t_1^4} = \frac{1-3t_1^2}{t_1^3-3t_1};$$

whence $4y(3t_1-t_1^3)+4x(1-3t_1^2)+at_1t_2(t_1^2-3)+a(1-3t_1^2)=0$,

or

$$y(3t_1-t_1^3)+x(1-3t_1^2)-2a=0, \text{ for } t_1t_2=3.$$

Thus P_1P_3 is the tangent at P_1; so is P_2P_4 at P_2.

The converse properties are very readily proved.

Eliminating t_1 and t_2 from the equations of the tangents at P_1 and P_2, and from $t^2+3=2tt'$, we get

$$2yt'^2+3xt'-3y=0 \text{ and } 3xt'^2+(3y+9x^2/2y)t'+\tfrac{1}{2}x-a=0;$$

whence, on eliminating t', we have

$$6y^2(15x^2+9y^2-3ax) = (10xy-2ay)^2 \quad \text{or} \quad 27y^2-5x^2=2a^2-11ax,$$

which is a hyperbola. Again, as $OO_1=-\frac{1}{4}a\sqrt{3}$, $OO_2=\frac{1}{4}a\sqrt{3}$, $OP_3=\frac{1}{4}at_2$, $OP_4=\frac{1}{4}at_1$, we get

$$\frac{O_1P_4}{P_4O_2} = \frac{t_1+\sqrt{3}}{\sqrt{3}-t_1} = \frac{\sqrt{t_2}+\sqrt{t_1}}{\sqrt{t_2}-\sqrt{t_1}} \text{ and } \frac{O_1P_3}{P_3O_2} = \frac{t_2+\sqrt{3}}{t_2-\sqrt{3}} = \frac{\sqrt{t_2}+\sqrt{t_1}}{\sqrt{t_2}-\sqrt{t_1}};$$

thus O_1O_2 is harmonically divided at P_3 and P_4.

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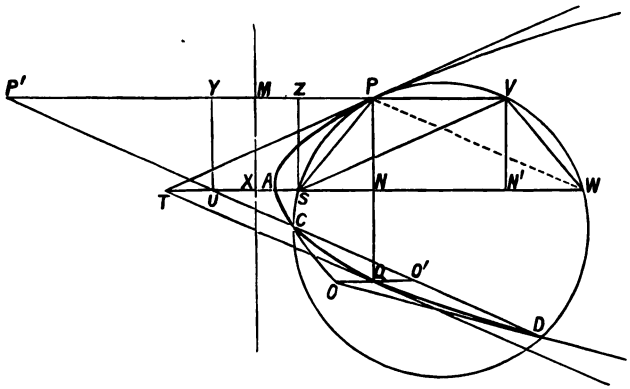
14535. (R. KNOWLES.)—A circle touches a parabola in a point P, and cuts it in C, D; prove (1) that the line joining the pole of CD, with respect to the parabola, to P is bisected by the axis; (2) if the circle pass through the focus, CD passes through a fixed point on the axis.

Solution by Professor SANJÁNA, M.A.; *and* W. J. GREENSTREET, M.A.

(1) The tangent at P and CD are equally inclined to the axis; if O be pole of CD, and OQO′ diameter, the tangent at Q is parallel to CD: hence the tangents at Q and P meet on axis, *i.e.*, PNQ is a double ordinate. Therefore, by similar triangles, OP is bisected by axis.



(2) $TP^2 = 4AN.ST$, by parabola, $= ST.TW$, by circle; hence $TW = 4AN$, *i.e.*, $AW = 3AN$. Therefore

$$PV = SW - SN - WN' = SW - 2SN, \text{ by circle,}$$
$$= AW - AS - 2SN = 2AN - SN = ST = SP = PM.$$

Again, if CD meet PM in P′, we have $P'C.P'D = P'P.P'V$, by circle, and $= 4P'P.SQ$, by parabola; hence $P'V = 4SP$ or $P'M = MV$. But SV is parallel to PT, as PV is equal and parallel to ST; hence SV and UP′ make equal angles with axis; therefore $YP' = VZ$, by equal triangles. Therefore the remainder $YM = MZ$, or $UX = SX$, *i.e.*, U is a fixed point on the axis.

14398. (D. BIDDLE.)—The sides AB, AC of a plane triangle ABC having been produced, a point P is taken between them, and a straight line is drawn through P and the mid-point of BC, cutting off one of the corners of the triangle. (1) Trace the boundary within which P must be placed so that a second line through it may cut off an equal triangular portion from ABC, at the other end of BC; (2) prove that, if P be restricted to the space between BC and a parallel intercept n times as long, the chance of satisfying the conditions is $2/(n+1)$; and (3) give a

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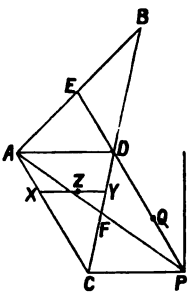
geometrical construction for the second line through P in the favourable cases.

Solution by H. W. Curjel, M.A.



(1) Let P be a point on the required boundary, and D the middle point of BC, and let PD cut AB in E. Then AP cuts off from the triangle ABC a triangle AFC equal to DEB.

Therefore E and F are equidistant from AD.

Therefore P and B are also equidistant from AD.

Therefore CP is parallel to AD; similarly, if E is in AC, PB is parallel to AD. Therefore, for a point to satisfy the given conditions, it is clear that it must lie between the parallels to AD through B and C.

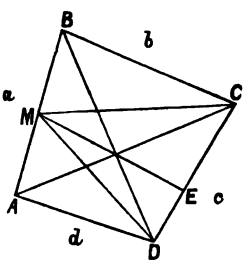
(2) The area between these parallels and BC and the parallel to it : the whole area between BC and the parallel to it and AC and BC produced $= 2 : n+1$.

Therefore required chance $= 2/(n+1)$.

(3) Let Q (in DP) be the point through which it is required to draw a second line; from CA, CB cut off CX, CY so that $CX^2 = CY^2 = CA.CB$; bisect XY in Z and describe a circle, centre C and radius CZ; draw tangents to it where it cuts CA, CB, and let these tangents meet in S; on QS as diameter describe a circle cutting the former circle in R, R'; then the one of QR and QR' which cuts off a part of triangle ABC is clearly the required straight line.

13282. (Professor Colart.) — Calculer, en fonction des côtés, le rapport des segments déterminés sur un côté d'un pseudocarré par la perpendiculaire abaissée du milieu du côté opposé.

Solution by W. J. Greenstreet, M.A.



Let $AC = BD = l$, and $AB = a$, &c. Draw ME perpendicular to CD from M, the mid-point of AB. Then

$$2MD^2 = d^2 + l^2 - \tfrac{1}{2}a^2,$$

$$2MC^2 = b^2 + l^2 - \tfrac{1}{2}a^2,$$

$$CE^2 - DE^2 = MC^2 - MD^2 = \tfrac{1}{2}(b^2 - d^2);$$

therefore $CE - DE = (b^2 - d^2)/2c$,

because $CE + DE = c$;

therefore $$\frac{CE + DE}{CE - DE} = \frac{2c^2}{b^2 - d^2};$$

therefore the ratio required

$$= \frac{CE}{DE} = \frac{2c^2 + b^2 - d^2}{2c^2 - b^2 + d^2}.$$

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14635. (R. F. Davis, M.A.)—Prove that (1) the trilinear coordinates of the foci of the ellipse touching the sides of the triangle of reference ABC at the feet of the perpendiculars are given by the equations $\beta/\gamma+\gamma/\beta = t\cos A$, $\gamma/\alpha+\alpha/\gamma = t\cos B$, $\alpha/\beta+\beta/\alpha = t\cos C$, where $t^3\cos A\cos B\cos C - t + 2 = 0$; (2) these foci all lie on the conics $\alpha^2\cot A+\beta\gamma\operatorname{cosec} A = \ldots = \ldots$; (3) F is a focus when, HL being drawn perpendicular upon AG (Quest. 13878), KF is taken a mean proportional between KA and KL, and bisects the angle between them. [G = centroid, H = orthocentre, K = symmedian point.]

Solution by Professor Sanjána, M.A.

If one *focus* be at the point (fgh), the equation of the inscribed ellipse is
$$\Sigma\left\{\alpha\left(gh^{-1}+hg^{-1}+2\cos A\right)\right\}^{\frac{1}{2}} = 0.$$
(See my Quest. 14289, *Ed. Times* for August, 1899.) Putting $\alpha = 0$, and remembering that, at the foot of the perpendicular AD, $\beta/\gamma = \cos C/\cos B$, we get $(h/f+f/h+2\cos B)\cos C = (f/g+g/f+2\cos C)\cos B$,
or $(h/f+f/h)\div\cos B = (f/g+g/f)\div\cos C$, $= (g/h+h/g)\div\cos A$, similarly.
If each of these is denoted by t, we have, since
$$(f/g+g/f)(g/h+h/g)(h/f+f/h) = (f/g+g/f)^2+(g/h+h/g)^2+(h/f+f/h)^2-4,$$
the following equation for t:—
$$t^3\cos A\cos B\cos C = -4+t^2(\cos^2 A+\cos^2 B+\cos^2 C)$$
$$= -4+t^2(1-2\cos A\cos B\cos C).$$
Therefore, supposing the factor $t+2$, we get (1)
$$t^2\cos A\cos B\cos C = t-2.$$
The equation of the ellipse is seen to be
$$\Sigma\left\{\alpha(t+2)\cos A\right\}^{\frac{1}{2}} = 0, \quad i.e., \quad \Sigma(\alpha\cos A)^{\frac{1}{2}} = 0.$$
On rationalizing, we get
$$\alpha^2\cos^2 A+\beta^2\cos^2 B+\gamma^2\cos^2 C-2\beta\gamma\cos B\cos C-2\gamma\alpha\cos C\cos A$$
$$-2\alpha\beta\cos A\cos B = 0.$$
Proceeding to find the focus of this conic, we obtain
$$(4bc\cos^2 A\cos B\cos C+4ca\cos^2 B\cos C\cos A+4ab\cos^2 C\cos A\cos B)\,\alpha^2$$
$$-4\Delta(2c\cos^2 B\cos C\cos A+2b\cos^2 C\cos A\cos B)$$
$$= (\ldots)\beta^2-4\Delta(\ldots)\beta = (\ldots)\gamma^2-4\Delta(\ldots)\gamma;$$
on simplifying, we see that $\alpha^2\cot\omega-a\alpha = \beta^2\cot\omega-b\beta = \gamma^2\cot\omega-c\gamma$, along with $a\alpha+b\beta+c\gamma = 2\Delta$, determine the coordinates of the foci. It is seen that these equations give four values of α, β, γ; thus we obtain, as usual, the two real and the two imaginary foci. Hence it is inaccurate to say that (2) the foci lie on certain conics; as a matter of fact, the conic $\Sigma(\alpha\cos A)^{\frac{1}{2}} = 0$ is quite determinate. If AK, BK, CK meet the circumcircle in A_1, B_1, C_1 respectively, the triangles LMN and $A_1B_1C_1$ are inversely similar, and K is their double point; hence A_1KL, B_1KM, C_1KN have the same bisectors. Consequently, the adjacent angles AKL, BKM, CKN have these same bisectors. This proves (3), and shows that the same pair of points is obtained by bisecting the angle between KA and KL as by bisecting the angle between KB and KM and that between KC and KN. [See Casey, 2nd ed., § 347.]

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14797. (Professor N. BHATTACHARYYA.)—A system of co-axial circles intersect in real points S, S′; two lines $SP_1P_2\ldots$, $SQ_1Q_2\ldots$ are drawn to intersect them in points $P_1, P_2, P_3, \ldots$; $Q_1, Q_2, Q_3, \ldots$. Show that the envelope of the chords $P_1Q_1, P_2Q_2, \ldots$ is a parabola whose focus is the point S′. If through any point T in the line SS′ a line be drawn to intersect the circles in A_1, A_2; B_1, B_2; C_1, C_2; &c., and if O be any point in the plane of the circles, show that the circles circumscribing the triangles OA_1A_2, OB_1B_2, ..., OSS′ will form a coaxial system.

Solution by R. F. DAVIS, M.A.

The foot of the perpendicular from S′ on P_1Q_1 lies on the line joining the feet of the perpendiculars from S′ on SP_1, SQ_1, which is a fixed straight line. Produce OT to O′, so that $OT \cdot O'T = ST \cdot S'T$; then the circles of the question all pass through a second fixed point O′.

14605. (D. BIDDLE.)—In a tetrahedron, the trilinear coordinates of the point of contact of the inscribed sphere with any face, taken in reference to the edges of that face, are in direct proportion to the combined perpendiculars to the respective edges, from the opposite vertex and its projection on the plane of the face, the ratio being that between the area of the particular face and the entire surface of the tetrahedron.

Solution by Rev. J. CULLEN.

Taking x, y, z, ω to represent the four planes, the equation of the inscribed sphere is

$$lqrx^2 + mpry^2 + npqz^2 + lmn\omega^2 + (lp - mq - nr)(lx\omega + pyz) + (mq - nr - lp)(my\omega + qxz) + (nr - lp - mq)(nz\omega + rxy) = 0$$

(SALMON, Art. 219, 2nd ed.), where $p = \cos^2 \tfrac{1}{2}(x\omega)$, $q = \cos^2 \tfrac{1}{2}(y\omega)$, $r = \cos^2 \tfrac{1}{2}(z\omega)$, $l = \cos^2 \tfrac{1}{2}(xy)$, $m = \cos^2 \tfrac{1}{2}(yz)$, $n = \cos^2 \tfrac{1}{2}(zx)$, putting $\omega = 0$, and observing that

$$x = a \sin(x\omega) = 2a \sqrt{\{p(1-p)\}}, \text{ &c.},$$

a being the perpendicular from the point of contact of the sphere with $\omega = 0$ (or ABC) on BC. We obtain the conic

$$\sqrt{(pqr)}\, l(1-p)\, \alpha^2 + \ldots + \ldots + \sqrt{\{p(1-q)(1-r)\}} \times (lp - mq - nr)\, \beta\gamma + \ldots + \ldots = 0,$$

which is clearly a circle of infinitely small radius, since it is the intersection of $\omega = 0$ with the sphere. Finding its centre in the ordinary manner, we obtain the trilinear coordinates of this point to be

$$a \sqrt{\{(1-p)/p\}} = \beta \sqrt{\{(1-q)/q\}} = \gamma \sqrt{\{(1-r)/r\}}$$

or
$$a \tan \tfrac{1}{2}(x\omega) = \beta \tan \tfrac{1}{2}(y\omega) = \gamma \tan \tfrac{1}{2}(z\omega).$$

But
$$\tan \tfrac{1}{2}(x\omega) = \sin(x\omega) / \{1 + \cos(x\omega)\}$$

and $\sin(x\omega) = \pi_\omega / p_a$, also $\cos(x\omega) = a_\omega / p_a$, where π_ω is the perpendicular from D on $\omega = 0$, p_a that on the edge a, and a_ω the projection p_a on $\omega = 0$. Hence the trilinears are proportional to $(p_a + a_\omega)$, &c.

Now $\Sigma \lambda a (p_a + a_\omega) = 2\Delta$, but $\Sigma a (p_a + a_\omega) = 2$ (entire surface of tetrahedron); therefore, &c.

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14673. (Rev. T. ROACH, M.A.)—A countryman, unversed in numbers, picks a bulrush, which he breaks into three pieces, and gives the longest to his eldest nephew Dick, the second to Sam, the next in age, and the third to Bob, the youngest. At the same time he promises each a ball of toffy as high as the length of his bulrush. Find the value of the expectation of each nephew, given that the whole bulrush was 24 inches long and that toffy costs $1\frac{1}{4}$d. a cubic inch, and taking the value of π as $3\frac{1}{7}$.

Solution by the PROPOSER.

Let the parts of the bulrush be α, $\alpha+\beta$, and $\alpha+\beta+\gamma$; then the value of the balls $= \frac{5}{4}\text{d.} \times \frac{4}{5} \times \frac{22}{7} \times (\frac{1}{3})^3 \times \Sigma\alpha^3$. Now $3\alpha+2\beta+\gamma = s = 24$ inches, and by WHITWORTH'S theorem of the expectation of parts we have, in the expansion of $(3\alpha+2\beta+\gamma)^3$,

$$\Sigma\alpha^p\beta^q\gamma^r = \frac{24^3 \times 2!}{5!} \frac{p!\,q!\,r!}{3^p 2^q},$$

where $p+q+r = n = 3$; therefore the value $= \frac{6}{7}\text{s.} \times \Sigma \frac{p!\,q!\,r!}{3^p 2^q}$,

and we require the values corresponding to α^3, $(\alpha+\beta)^3$, and $(\alpha+\beta+\gamma)^3$. Hence the values $= \frac{3}{7}\text{s.} \times \frac{3}{8}, \frac{3}{7}\text{s.} \times (\frac{3}{8}+\frac{1}{3}+\frac{1}{3}+\frac{3}{4})$, and $\frac{3}{7}\text{s.} \times (\frac{9}{8}+\frac{1}{6}+5+6)$

$= 2\text{s. } 9\frac{1}{11}\text{d.}$, £1. 2s. $8\frac{8}{11}$d., and £10. 0s. $9\frac{1}{11}$d.

[Prebendary WHITWORTH'S theorem was published in a pamphlet three years ago, and is incorporated into the 5th edition of *Choice and Chance*, now in the press. It is as follows:—"If n random magnitudes $\alpha, \beta, \gamma, \ldots$ be subject only to the condition $a\alpha+b\beta+c\gamma+\ldots = s$, then every term in the expansion of $(a\alpha+b\beta+c\gamma+\ldots)^m$ is of equal expectation, m being any positive integer or fraction."]

14684. (D. BIDDLE.)—If a tetrahedron have its opposite edges equal, the point of contact of the inscribed sphere with any face is at the circumcentre of the face.

Solution by JAMES BLAIKIE, M.A.; Rev. J. CULLEN; *and many others.*

Since the opposite edges are equal, the four triangular faces are congruent, by *Euclid* I. 8. Therefore their circumscribing circles are equal; that is, their planes cut the circumscribing sphere in equal circles, and are therefore equally distant from its centre. This point is therefore also the centre of the inscribed sphere, and its points of contact, being the feet of the perpendiculars from the centre of the circumscribing sphere, are the centres of the various circumscribing circles.

[Mr. W. J. GREENSTREET, M.A., observes:—The tetrahedron whose opposite edges are equal in pairs is called the isosceles tetrahedron. A large number of properties of this tetrahedron will be found in the *Nouvelles Annales*, 1880, pp. 133, 403; also *v.* CAYLEY, *Collected Papers*, Vol. v., p. 550.

Mr. GREENSTREET further gives the following interesting properties of the "isosceles" tetrahedron T:—

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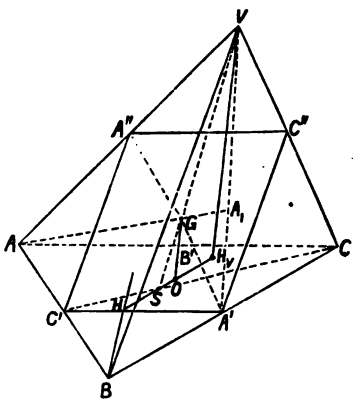

(1) The four faces are equal. (I. 8.)

(2) The four perpendiculars of T are equal (because the bases are equal).

(3) The opposite dihedral angles are equal, for the trihedral angles are all equal, the faces being equal each to each, and the homologous dihedrals are those which have as edges the opposite edges of T.

(4) The six bisecting planes of the dihedral angles cut in the centre of gravity of T. Bisect the dihedral VA by the plane AVA'. Then AVA' bisects BC in A', *i.e.*, it coincides with the plane through VA and the mid-point of the opposite side BC.

(5) The four joins, such as AA_1, from a vertex to the centre of gravity of the opposite face are concurrent in G, the centre of in- and circum-spheres of T, and are equal. For AA_1, VS intersect in G, such that $GS = \frac{1}{4}SV$, and therefore G is on the four lines, and is the centre of gravity of T. By (4) it is also on the bisecting planes, and is therefore the incentre of T, and the inradius is thus $\frac{1}{4}$ of the height VH_V. The medians AA', VA' of two equal faces are equal; therefore $\triangle AVA'$ is isosceles; therefore AVA_1S is isosceles, for $AS = VA_1 = \frac{3}{4}AA'$, and, its diagonals being equal, $AA_1 = VS$; therefore $AA_1 = BB_1 = CC_1 = VS$; $GA = \frac{3}{4}AA_1$; $GV = \frac{3}{4}VS$; therefore $GA = GB = GC = GV$; therefore G is circumcentre of T.

(6) The mid-joins A'A'', C'C'', &c., of opposite edges are the common perpendiculars respectively of the edges, are perpendicular in pairs, and concurrent in G. For $A'A = A'V$, $GA = GV$; therefore A'G is perpendicular to AV, and is the perpendicular A'A'' of the isosceles triangle AVA', and A'' is mid-point of AV. Hence (as in any tetrahedron) A'A'', the join of mid-points of BC and AV, passes through G; and is perpendicular to AV and BC. Similarly for C'C''. Thus A'C'A''C'' is a rhombus, for $A'C' = A''C'' = \frac{1}{2}AC$, $C'A'' = A'C'' = \frac{1}{2}BV$, and $AC = BV$; therefore A'A'' and C'C'' intersect at G at right angles.

(7) H_V, S, O are collinear, when VH_V is perpendicular to plane ABC, S is its centre of gravity, and O its circumcentre. Let a, b, c be the edges of T. Then, from the right-angled triangles VA'A'' and GAA'', we have

$$A'A''^2 = VA'^2 - VA''^2 = \tfrac{1}{4}(b^2 + c^2 - a^2), \text{ \&c.}$$

So the rhombus A'C'A''C'' gives $GA'' = \frac{1}{2}A'A''$; therefore

$$GA''^2 = \tfrac{1}{8}(b^2 + c^2 - a^2), \quad GA^2 = AA''^2 + GA''^2 = R^2 = \tfrac{1}{8}\Sigma a^2;$$

therefore $GA = GB = GC$; therefore, GO being the perpendicular on ABC, $OA = OB = OC$. Then O is circumcentre of ABC, and H_VSO is the EULER-line of ABC. $VS = 4SG$; therefore $SH_V = 4SO$, $OH_V = 3SO$.

(8) If H is orthocentre of ABC, O bisects HH_V. For OSH_V is the EULER-line of ABC, and $SH = 2SO$; therefore $OH = 3SO = OH_V$, by (7).]

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14369. (G. H. HARDY, B.A.—If $\sum_0^\infty u_n(x)$ be a *summable divergent* series, having the sum $\int_0^\infty e^{-\alpha} u(x, \alpha)\, d\alpha$,

where $$u(x, \alpha) = \sum_0^\infty \frac{\alpha^n}{n!} u(x),$$

prove that it may be integrated term by term in (x_0, X), provided (1) $u(x, \alpha)$ may be integrated term by term, and (2) the order of integration in $$\int_0^\infty e^{-\alpha} d\alpha \int_{x_0}^X u(x, \alpha)\, dx$$ may be changed. Deduce and verify independently that, if

$$\phi(x, \alpha) = e^{-\alpha(1+\cos x)} \cos(x - \alpha \sin x),$$
$$\psi(x, \alpha) = e^{-\alpha(1-\cos 2mx)} \cos(mx + \alpha \sin 2mx)/(1 + x^2),$$

then $$\int_0^{2\pi} dx \int_0^\infty \phi\, d\alpha = \pi, \quad \int_0^\infty d\alpha \int_0^{2\pi} \phi\, dx = 0;$$
$$\int_0^\infty dx \int_0^\infty \psi\, d\alpha = 0, \quad \int_0^\infty d\alpha \int_0^\infty \psi\, dx = \tfrac{1}{2} \frac{\pi}{e^m - e^{-m}}.$$

Solution by the PROPOSER.

According to BOREL's definition of the sum of a divergent series,

$$\sum_0^\infty u_n(x) = \int_0^\infty e^{-\alpha} u(x, \alpha)\, dx.$$

Hence $$\int_{x_0}^X dx \sum_0^\infty u_n(x) = \int_{x_0}^X dx \int_0^\infty e^{-\alpha} u(x, \alpha)\, dx = \int_0^\infty e^{-\alpha} d\alpha \int_{x_0}^X u(x, \alpha)\, dx$$
$$= \int_0^\infty e^{-\alpha} d\alpha \int_{x_0}^X \sum_0^\infty \frac{\alpha^n}{n!} u_n(x)\, dx$$
$$= \int_0^\infty e^{-\alpha} d\alpha \sum_0^\infty \frac{\alpha^n}{n!} \int_{x_0}^X u_n(x)\, dx = \sum_0^\infty \int_{x_0}^X u_n(x)\, dx;$$

provided conditions (1), (2) be satisfied. Here X may be finite or infinite. It follows that, if integration of the divergent series term by term leads to a false result, one or other of (1), (2) cannot be satisfied. Generally it is (2).

Thus consider the divergent series $\cos x - \cos 2x \ldots$.

In this case $$u(x, \alpha) = \sum_0^\infty (-)^n \frac{\alpha^n}{n!} \cos(n+1)x = e^{-\alpha \cos x} \cos(x - \alpha \sin x),$$

and $$\int_0^\infty e^{-\alpha} u(x, \alpha)\, d\alpha = \frac{\cos x (1 + \cos x) + \sin^2 x}{(1 + \cos x)^2 + \sin^2 x} = \tfrac{1}{2};$$

so that $$\cos x - \cos 2x + \ldots = \tfrac{1}{2}.$$

If we integrate from 0 to 2π, we find $0 = \pi$.

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Now it is easy to see that in this case (1) is satisfied. If follows that, if

$$\phi(x, a) = e^{-a(1+\cos x)} \cos(x - a \sin x),$$

$$\int_0^{2\pi} dx \int_0^{\infty} \phi \, da = \pi, \qquad \int_0^{\infty} da \int_0^{2\pi} \phi \, dx = 0.$$

The second equation is easily verified, for

$$\int_0^{2\pi} \phi \, dx = -\frac{e^{-a}}{a} \left(e^{-ae^{ix}} \right)_0^{2\pi} = 0.$$

Again, it is easy to show that $\cos mx + \cos 3mx + \ldots = 0$.

Multiply by $\frac{1}{1+x^2}$, and integrate from 0 to ∞. We find

$$\tfrac{1}{2} \frac{\pi}{e^m - e^{-m}} = 0;$$

and it follows that, if

$$\psi(x, a) = e^{-a(1-\cos 2mx)} \frac{\cos(mx + a \sin 2mx)}{1+x^2},$$

$$\int_0^{\infty} dx \int_0^{\infty} \psi \, da = 0, \qquad \int_0^{\infty} da \int_0^{\infty} \psi \, dx = \tfrac{1}{2} \frac{\pi}{e^m - e^{-m}}.$$

The second equation is also easily verified, for

$$\int_0^{\infty} \psi \, dx = e^{-a} \int_0^{\infty} \frac{dx}{1+x^2} \sum_0^{\infty} \frac{a^n}{n!} \cos(2n+1) mx$$

$$= \tfrac{1}{2}\pi e^{-a} \sum_0^{\infty} \frac{a^n}{n!} e^{-(2n+1)m} = \tfrac{1}{2}\pi e^{-m-a(1-e^{-2m})};$$

and

$$\tfrac{1}{2}\pi e^{-m} \int_0^{\infty} e^{-a(1-e^{-2m})} \, da = \tfrac{1}{2} \frac{\pi}{e^m - e^{-m}}.$$

14644. (V. Daniel.)—If a chord of the parabola $y^2 = 4ax$ and the tangents at its extremities form a triangle of constant area $\frac{1}{2}\lambda^3 a^2$, the pole of the chord lies on the parabola $y^2 = 4ax + \lambda^2 a^2$.

Solution by I. Nisner; V. Ragunath Rau, B.A.; *and* W. J. Greenstreet, M.A.

Let the coordinates of the extremities of the chord be $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$, where t_1 and t_2 are respectively the cotangents of the angles which the tangents at the extremities of the chord make with the axis. The coordinates of the point of intersection of the tangents will be $\{at_1t_2, a(t_1+t_2)\}$. The area of the triangle so formed will be

$$\tfrac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ 2at_1 & 2at_2 & a(t_1+t_2) \\ at_1^2 & at_2^2 & at_1t_2 \end{vmatrix} = \tfrac{1}{2}a^2(t_1-t_2)^2 \begin{vmatrix} 0 & 1 & 0 \\ 2 & 2t_2 & 1 \\ t_1+t_2 & t_2^2 & t_2 \end{vmatrix} = \tfrac{1}{2}a(t_1-t_2)^3 = \tfrac{1}{2}a^2\lambda^3;$$

therefore $(t_1-t_2)^2 = \lambda^2$.

Again, if $X = at_1t_2$ and $Y = a(t_1+t_2)$,

$$Y^2 = a^2(t_1+t_2)^2 = 4a^2t_1t_2 + a^2(t_1-t_2)^2 = 4aX + a^2\lambda^2.$$

Hence, &c.

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